

REPORT

Influence of ohmic contact loss on measurement of dielectric properties of ultra-low-loss dielectrics

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ABSTRACT

The effect of plate ohmic contacts attached to a low-loss dielectric substrate on the measured dielectric constant and loss tangent is investigated. To this end the effect of longitudinal and transverse currents and charge pile-up in these contacts is analysed. Three types of parallel-plate capacitor are investigated: one-dimensional, circular disk and interdigital. For all three capacitor type, closed-form expressions are derived for the expectancy value of the overall tangent of the dielectric with plated contacts. A systematic set of numerical examples is presented for each capacitor type, demonstrating in detail the dependencies of the loss tangent on the various parameters. A procedure for experimental verification of the theoretical results is proposed.

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1 Introduction

In the measurement of ultra-low-loss dielectric substrates, the effect of losses caused by the ohmic contact plates of the electrodes may significantly impair the accuracy of computations in the inversion procedure for the complex permittivity of the substrate. Therefore, a thorough understanding and quantification of this effect is essential in deriving an inversion procedure that annihilates the effect of these electrodes. The present study tackles this problem on a step-by-step basis for various types of capacitors.

2 One-dimensional capacitor

2.1 Introduction

As a simple example, consider a 1D capacitor, of infinite extent in the direction perpendicular to the page. Fig 1 shows the configuration of interest.

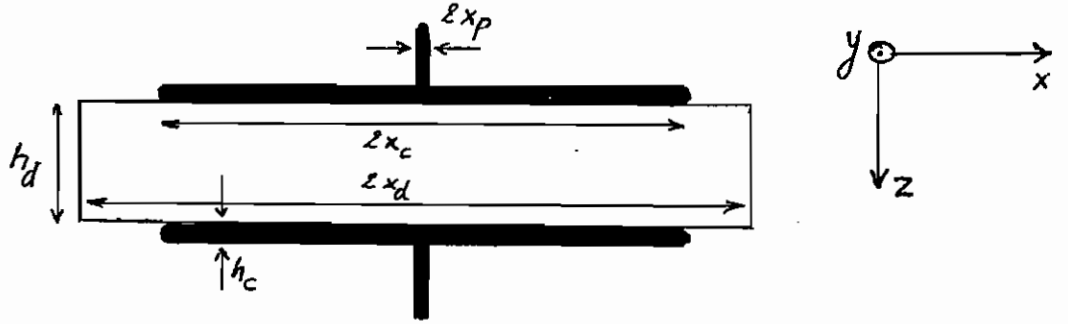


Figure 1: One-dimensional capacitor.

2.2 Longitudinal electron motion and interfacial polarisation

2.2.1 Introduction

Current flowing in the direction of the axis of the measurement configuration causes ohmic losses when traversing the contact (plating) layers. In addition, the piling-up of free electrons at the interface between the ohmic contact and the dielectric causes interfacial and space-charge polarisation to occur [1].

The first question to be addressed is determining the area of the conductor (measured in transverse direction) across which the effect exists. At first, it would occur that this area would be only the area defined by the probe cross-section, because underneath the probe the motion of electrons is purely longitudinal causing the pile-up of charges. However, in the quasi-static approximation all planes of constant height z ($0 \leq z \leq h_c$) are equipotential planes, hence the electrons in such planes move in z -direction in a synchronized manner. Thus, all electrons in an *entire* cross-section of the conducting plate are 'flushed' towards or away from both metal/dielectric interfaces. Therefore the effective area A of the Maxwell-Wagner condenser [1] is $2x_c y_c$ and not $2x_p y_c$, provided $x_d \geq x_c$.

2.2.2 Effect on measured complex permittivity

An analysis of a Maxwell-Wagner two-layer condenser can be found in [1]. This leads to the introduction of an effective complex permittivity $\epsilon = \epsilon' - j\epsilon''$ of the condenser:-

$$\frac{\epsilon'}{\epsilon_o} = \frac{\tau_c + \tau_d - \tau + \omega^2 \tau_c \tau_d \tau}{C_o(R_{cl} + R_d)(1 + \omega^2 \tau^2)} \quad (1)$$

$$\frac{\epsilon''}{\epsilon_o} = \frac{1 + \omega^2 [\tau(\tau_c + \tau_d) - \tau_c \tau_d]}{\omega C_o(R_{cl} + R_d)(1 + \omega^2 \tau^2)} \quad (2)$$

i.e.

$$\tan \delta \triangleq \frac{\epsilon''}{\epsilon'} = \frac{1 + \omega^2 [\tau(\tau_c + \tau_d) - \tau_c \tau_d]}{\omega (\tau_c + \tau_d - \tau + \omega^2 \tau_c \tau_d \tau)} \quad (3)$$

with

- $\tau_c = R_{cl}C_c$: time constant of equivalent circuit for contact plate layers in longitudinal direction;
- $\tau_d = R_dC_d$: time constant of equivalent circuit for dielectric layer under test (DUT);
- $\tau = RC = \frac{R_{cl}R_d(C_c+C_d)}{R_{cl}+R_d} = \frac{R_{cl}\tau_d+R_d\tau_c}{R_{cl}+R_d} = \frac{\epsilon'_c h_d + \epsilon'_d h_c}{\sigma_c h_d + \sigma_d h_c}$: time constant of equivalent circuit for condenser;
- $R_{cl} = h_c/(x_c y_c \sigma_c)$: equivalent circuit resistance for contact plate layers;
- $C_c = x_c y_c \epsilon'_c / h_c$: equivalent circuit capacitance for contact plate layers;
- $R_d = h_d/(x_c y_c \omega \epsilon''_d)$: equivalent circuit resistance for DUT (note: expression contains x_c , not x_d !);
- $C_d = x_c y_c \epsilon'_d / h_d$: equivalent circuit capacitance for DUT (note: expression contains x_c , not x_d !);
- $C_o = x_c y_c \epsilon_o / (h_d + 2h_c)$: capacitance of a vacuum condenser.

We consider a condenser in which the outer 'dielectric' layers are good¹ conductors ($R_{cl} \ll R_d$). Then $\tau \approx \tau_c + R_{cl}C_d$. After substitution into Eqn (3), we obtain:-

$$\tan \delta_T \approx \frac{1 + \omega^2 [R_{cl}R_dC_d^2 + R_{cl}^2C_c(C_c + C_d)]}{\omega [R_dC_d - R_{cl}C_d + \omega^2 R_{cl}^2C_cR_dC_d(C_c + C_d)]} \quad (4)$$

The following approximations can be made:-

¹relative to the conductivity/polarisation loss of the DUT

- *0th order* ($R_{cl}/R_d \rightarrow 0$): Multiplying numerator and denominator by R_{cl}^{-2} and neglecting all terms in R_{cl}/R_d , we retrieve:-

$$\tan \delta_T \approx (\omega R_d C_d)^{-1} = \tan \delta_d \quad (5)$$

i.e. the overall loss tangent is governed by the loss tangent of the DUT only, as expected;

- *to 1st order* ($R_{cl}/R_d \ll 1$): Multiplying numerator and denominator by R_{cl}^{-2} , neglecting all second-order terms in R_{cl}/R_d and using the approximation $(1-x)^{-1} \approx 1+x$ for $x \ll 1$, we obtain:-

$$\tan \delta_T \approx \frac{R_d^{-2} + \omega^2 (R_{cl}/R_d) C_d^2}{\omega (1 - R_{cl}/R_d) (C_d/R_d)} \approx \tan \delta_d + \frac{R_{cl}}{R_d} [\tan \delta_d + (\tan \delta_d)^{-1}] \quad (6)$$

For a measured value $\tan \delta_T = \epsilon''/\epsilon'$, this yields, to first approximation², a quadratic equation with two solutions, of which we select the solution with the negative sign, for this one correctly reduces to zero if $R_{cl} \rightarrow 0$:-

$$\tan \delta_d \approx \frac{\tan \delta_T}{2 \left(1 + \frac{R_{cl}}{R_d}\right)} \left[1 - \sqrt{1 - \frac{4R_{cl}(R_{cl} + R_d)}{R_d^2 \tan^2 \delta_T}} \right] \quad (7)$$

- *further higher-order approximations*: these can be made for cases where R_{cl}/R_d is not very small. This results in a power series in (R_{cl}/R_d) .

Because quasi-static excitation is assumed, the sequence of the layers is irrelevant to the calculation of the effective properties of the condenser. Therefore, the top and bottom contact plates may be collected into a single layer of twice the single-layer thickness, with associated capacitance $C_c/2$ and resistance $2R_{cl}$.

2.3 Transverse electron motion

2.3.1 Mechanism

Since the electric field is directed perpendicular to the conducting plates, it cannot generate a transverse current by itself. Instead, a self-induced transverse current can flow as a result of the Lorentz law:-

$$\underline{F} = q(\underline{E} + \underline{v} \times \underline{B}) \quad (8)$$

Indeed, for N charge units per unit volume a z -directed primary current $N\Delta V q \underline{v} = N\Delta V q v \underline{1}_z$ is induced in the conduction plates by $\underline{E} = E \underline{1}_z$. This current induces a magnetic field such

²Note that R_d is implicitly dependent on ϵ_d and $\tan \delta_d$.

that $\nabla \times \underline{H}_{induced} = N\Delta V q \underline{v}$ with the curl acting within the transverse plane. Hence the magnetic force on the electrons travelling in z -direction is:-

$$\begin{aligned}\underline{F}_m &= N\Delta V q \underline{v} \times \underline{B} = \mu_o (\nabla \times \underline{H}_{induced}) \times \underline{H} \\ &= -\mu_o \nabla (\underline{H} \cdot \underline{H}_{induced})\end{aligned}\quad (9)$$

which is nonzero because both the primary and induced magnetic fields are transverse. The minus sign being a result of the Lenz rule. Hence a transverse force is exerted on the travelling electrons, causing a transverse induction current. Note that, as an immediate consequence of the self-induction, it follows that if the plate is arbitrarily thin to such extent that it reduces to a single-electron layer, no longitudinal and thus no induced transverse currents exist.

2.3.2 Effect on measured complex dielectric constant

For the transverse current, the equivalent circuit of Fig 2 applies.

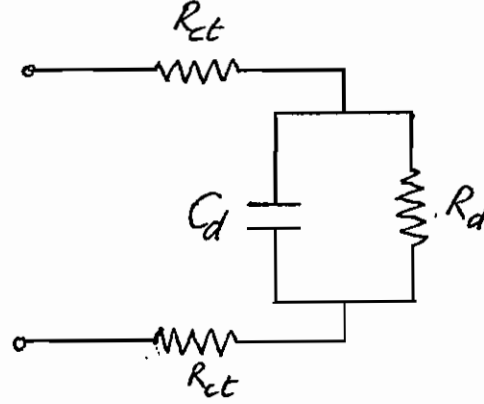


Figure 2: Equivalent circuit for transverse current in plates of one-dimensional capacitor.

The input impedance of this circuit is:-

$$Z = 2R_{ct} + (R_d^{-1} + j\omega C_d)^{-1} \quad (10)$$

$$= \frac{2R_{ct} + R_d + j\omega 2R_d C_d R_{ct}}{1 + j\omega R_d C_d} \quad (11)$$

which corresponds to an admittance:-

$$Y = Z^{-1} = \frac{2R_{ct} + R_d + 2\omega R_d^2 C_d^2 R_{ct} + j\omega R_d^2 C_d}{(2R_{ct} + R_d)^2 + (2\omega R_d C_d R_{ct})^2} \quad (12)$$

hence the loss tangent $\tan \delta_T$ of the total system, i.e. DUT + two ohmic contact plates, which includes dielectric losses and losses due to transverse currents (but not longitudinal currents), is given by:-

$$\tan \delta_T = \frac{Re(Y)}{Im(Y)} = \frac{R_d^{-1} + 2R_{ct} [R_d^{-2} + (\omega C_d)^2]}{\omega C_d} = \frac{1 + 2R_{ct}R_d^{-1} [1 + (\omega C_d R_d)^2]}{\omega C_d R_d} \quad (13)$$

In these expressions, the factor '2' is due to the fact that both the top and bottom contact plates are to be taken into account. The various symbols refer to:-

- $C_d = \epsilon' x_c y_c / h_d$: equivalent network capacitance of the DUT;
- $R_d = h_d / (\omega \epsilon'' x_c y_c)$: equivalent network resistance of the DUT;
- $R_{ct}(x - x_p) \approx (x - x_p) / (y_c h_c \sigma_c)$: equivalent network resistance³ of one half of one conducting plate across a distance $x - x_p$ ($x_p \leq x \leq x_c$) in the transverse direction;
- $\epsilon = \epsilon' - j\epsilon''$: complex permittivity of the DUT;
- $\sigma_c = \rho_c^{-1}$: surface conductivity of the conducting plating material;
- $2x_c, y_c, h_c$: dimensions of one conducting plating layer;
- h_d : height of the DUT;
- $2x_p$: thickness of the probe wire.

It is verified that in the limit $R_{ct} \rightarrow 0$, Eqn (13) reduces to the loss tangent of the dielectric only, $\tan \delta_d = (\omega R_d C_d)^{-1} = \epsilon'' / \epsilon'$. Integration yields the expectancy value $\langle R_{ct}(x - x_p) \rangle$:-

$$\langle R_{ct}(x - x_p) \rangle = 2 \frac{\int_0^{y_c} dy \int_{x_p}^{x_c} R_{ct}(x - x_p) d(x - x_p)}{\int_0^{y_c} dy \int_{x_p}^{x_c} d(x - x_p)} = \frac{x_c - x_p}{y_c h_c \sigma_c} \quad (14)$$

The factor '2' in Eqn (14) is due to the fact that the two halves of the plate, on both sides of the probe, are to be used with the definitions of C_d and R_d given above, because the current flows symmetrically with respect to $x = 0$.

The expectancy value for the overall loss tangent is therefore:-

$$\begin{aligned} \langle \tan \delta_T \rangle &= \frac{R_d^{-1} + 2\langle R_{ct} \rangle R_d^{-2} [1 + (\omega R_d C_d)^2]}{\omega C_d} \\ &= \frac{\epsilon''}{\epsilon'} + \frac{2x_c(x_c - x_p)}{h_c h_d} \frac{\omega \epsilon'}{\sigma_c} \left[1 + \left(\frac{\epsilon''}{\epsilon'} \right)^2 \right] \end{aligned} \quad (15)$$

$$\triangleq \tan \delta_d + \omega K_1 C_d \quad (16)$$

with

³The expression for R_{ct} assumes a uniform transverse current density in the conducting plate. Since in reality the value of x_c is finite, this density reaches a maximum underneath the probe and zero at the plate edges. Hence the expression for R_{ct} is only approximate (worse for small x_c). A more accurate analysis requires knowledge of the detailed transverse current distribution.

$$\tan \delta_d = \frac{\epsilon''}{\epsilon'} \quad (17)$$

$$K_1 = \frac{2(x_c - x_p)}{y_c h_c \sigma_c} [1 + \tan^2 \delta_d] \quad (18)$$

Thus, the overall loss tangent consists of two terms: the loss tangent of the DUT and a contribution by both the ohmic contact plate and DUT. If the transverse dimension in x -direction is much larger than the probe thickness, i.e. $x_p \ll x_c$, then the second term in the expression for $\tan \delta_T$ is seen to increase with approximately the square of x_c . It is verified that $\tan \delta_T$ is independent of y_c , as expected.

The consideration of transverse currents is heuristic, in the sense that no impressed sources of transverse current (no transverse electric field gradient nor a normal magnetic field component H_z) exist in the quasi-static approximation. The voltage is applied between the probes, i.e. in z -direction. Therefore, in the quasi-static approximation, all free electrons in a plane $z = \text{constant}$ react instantaneously and exhibit the same potential (energy level). Of course, realistic metals used for the contact plates show a *non-zero* response time associated with the transverse dimension of the contact⁴. This 'transverse-response' time, τ_t gives rise to transverse *transient* currents, in order to convey the field to all electrons in each z -plane. As a result, the associated loss tangent $\tan \delta_t$ manifests itself at operating frequencies of the order of the transverse-resonance frequency ω_t :

$$\omega_t = (\tau_t)^{-1} = \frac{2\pi v}{\lambda_{res}} = \frac{\pi v}{x_c} \quad (19)$$

The estimation of the wave velocity, v , is difficult because the transverse wave is a surface wave. In the MHz range of operating frequencies ω and with $v = c$, this requires x_c to be of the order of several metres for it to show a significant contribution to the overall loss: in this case, the second term in the expression for $\tan \delta_T$ becomes of the order of:-

$$\frac{2\pi v x_c (x_c - x_p)}{h_c h_d} \frac{\omega \epsilon'}{\sigma_c} [1 + \tan^2 \delta_d] \quad (20)$$

This term must be of the order of $\tan \delta_d$ for it to have a significant effect on $\langle \tan \delta_T \rangle$ in quasi-static regime.

2.4 Total electron motion

Substituting Eqn (3) into Eqns (16)–(18) the overall loss tangent, containing contributions of dielectric loss and transverse and longitudinal current resistive losses in the contact plates, is obtained.

⁴This response time is itself orders of magnitude larger than the response of the individual electrons, whose effect can certainly be neglected.

The above treatment is approximate because the path of the electrons is approximated by a transverse motion (parallel to the plate surfaces), followed by a longitudinal motion (i.e. perpendicular to the plate surfaces (z direction)). In this way the actual incremental electron displacement $d\mathbf{R}(r, h)$ at a local transverse distance r and local depth h is approximated by

$$dR \cos\theta(r, h) \mathbf{1}_x + dR \sin\theta(r, h) \mathbf{1}_z \quad (21)$$

where $\theta(r, h)$ determines the direction of the tangent to local path of the electron in the conducting plate at (r, h) . This approximate path has a length $dR[\cos\theta(r, h) + \sin\theta(r, h)]$ which is always larger than (or at most — for $\theta(r, h) = 0$ or 90 deg — equal to) the actual path length, dR . Thus, the loss tangent computed with the above formulas is an upper limit for the actual (measured) loss tangent of the complete system. A more accurate calculation requires a detailed knowledge of the path of the electrons at each position within the contact plate, i.e. $\theta(r, h)$.

2.5 Numerical results

A number of numerical simulations for the above model have been performed. The effective permittivity $\epsilon = \epsilon' - j\epsilon''$ of the condenser is computed first, from input data (estimates) for the individual layers, $\epsilon'_d - j\epsilon''_d$ and $\epsilon'_c - j\sigma_c/(\omega\epsilon_o)$. This ϵ is then used to compute losses due to transverse currents, resulting in a value $\tan\delta_T$ for the overall loss.

Figs 3–8 show the dependency of the overall loss tangent $\tan\delta_T$ as a function of various variables for several constant parameter values:-

- Fig 3: $\tan\delta_T(f)$ for selected values of x_c ;
- Fig 4: $\tan\delta_T(f)$ for selected values of x_p ;
- Fig 5: $\tan\delta_T(x_c)$ for selected values of f ;
- Fig 6: $\tan\delta_T(h_c)$ for selected values of f ;
- Fig 7: $\tan\delta_T(h_c)$ for selected values of x_c ;
- Fig 8: $\tan\delta_T(\tan\delta_d)$ for selected values of f .

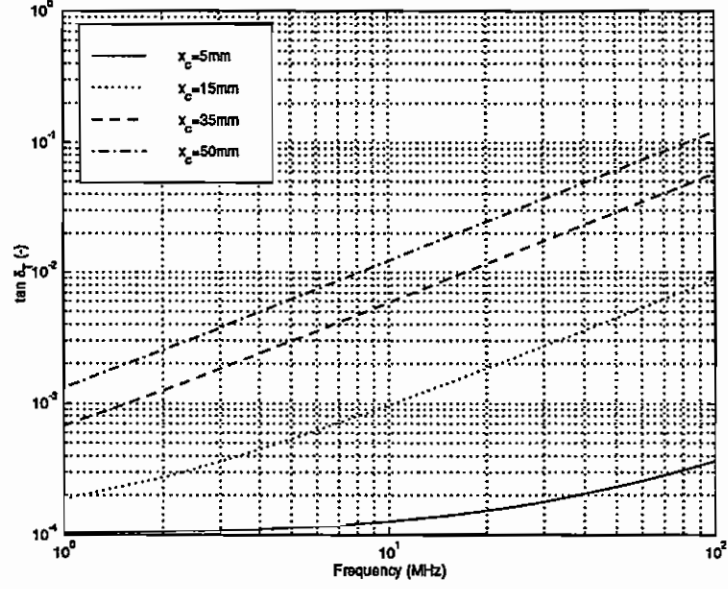


Figure 3: $\tan \delta_T$ as a function of f for $x_c = 5\text{mm}$, 15mm , 35mm and 50mm . $x_p = 4\text{mm}$, $h_c = 0.1\text{ }\mu\text{m}$, $h_d = 1\text{mm}$, $\epsilon'_d/\epsilon_o = 10$, $\tan \delta_d = 10^{-4}$, $\sigma_c = 2.1 \times 10^7\text{ S/m}$.

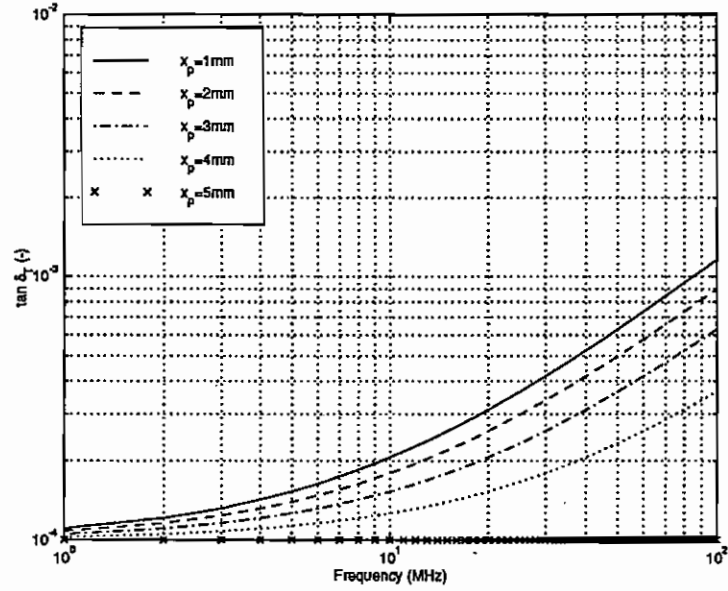


Figure 4: $\tan \delta_T$ as a function of f for $x_p = 1\text{mm}$, 2mm , 3mm , 4mm and 5mm . $x_c = 5\text{mm}$, $h_c = 0.1\text{ }\mu\text{m}$, $h_d = 1\text{mm}$, $\epsilon'_d/\epsilon_o = 10$, $\tan \delta_d = 10^{-4}$, $\sigma_c = 2.1 \times 10^7\text{ S/m}$.

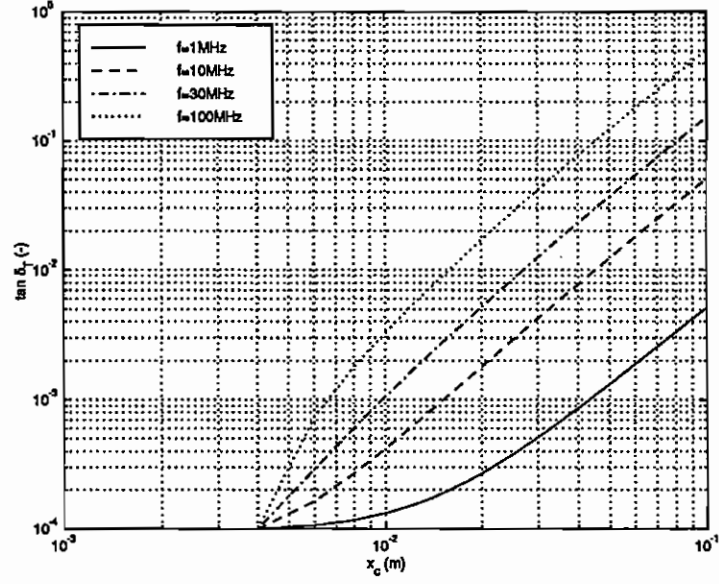


Figure 5: $\tan \delta_T$ as a function of x_c for $f = 1\text{MHz}$, 10MHz , 30MHz and 100MHz . $x_p = 4\text{mm}$, $h_c = 0.1\text{ }\mu\text{m}$, $h_d = 1\text{mm}$, $\epsilon'_d/\epsilon_o = 10$, $\tan \delta_d = 10^{-4}$, $\sigma_c = 2.1 \times 10^7\text{ S/m}$.

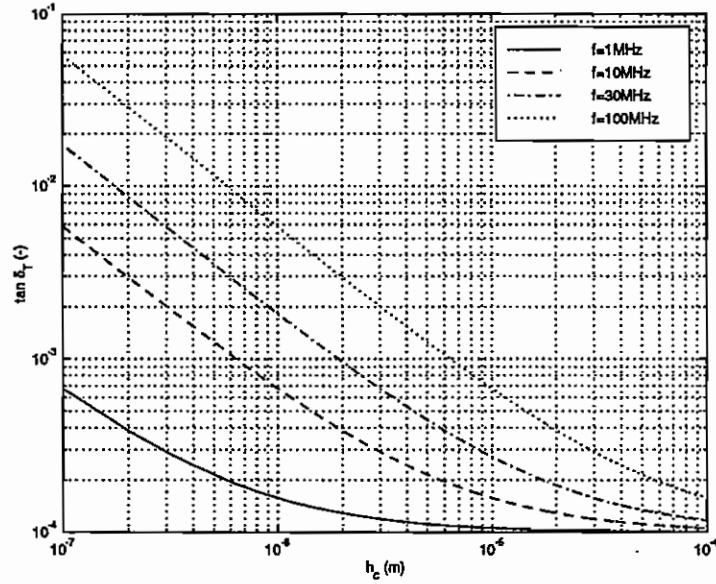


Figure 6: $\tan \delta_T$ as a function of h_c for $f = 1\text{MHz}$, 10MHz , 30MHz and 100MHz . $x_c = 35\text{mm}$, $x_p = 4\text{mm}$, $h_d = 1\text{mm}$, $\epsilon'_d/\epsilon_o = 10$, $\tan \delta_d = 10^{-4}$, $\sigma_c = 2.1 \times 10^7\text{ S/m}$.

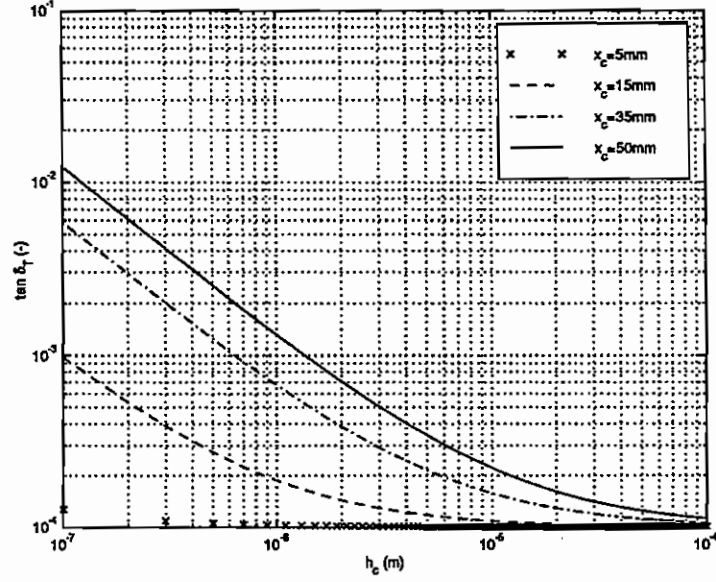


Figure 7: $\tan \delta_T$ as a function of h_c for $x_c = 5\text{mm}$, 15mm , 35mm , 50mm . $f = 30\text{MHz}$, $x_p = 4\text{mm}$, $h_d = 1\text{mm}$, $\epsilon'_d/\epsilon_o = 10$, $\tan \delta_d = 10^{-4}$, $\sigma_c = 2.1 \times 10^7 \text{ S/m}$.

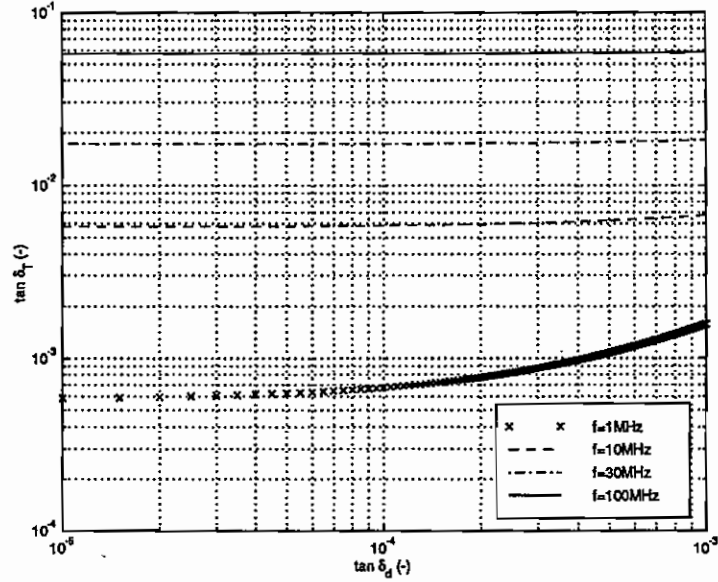


Figure 8: $\tan \delta_T$ as a function of $\tan \delta_d$ for $f = 1\text{MHz}$, 10MHz , 30MHz and 100MHz . $x_c = 35\text{mm}$, $x_p = 4\text{mm}$, $h_c = 0.1\text{mm}$, $h_d = 1\text{mm}$, $\epsilon'_d/\epsilon_o = 10$, $\sigma_c = 2.1 \times 10^7 \text{ S/m}$.

3 Circular disk capacitor

3.1 Introduction

A more practical example of a parallel-plate capacitor is the circular disk capacitor. Fig 9 shows the configuration of interest.

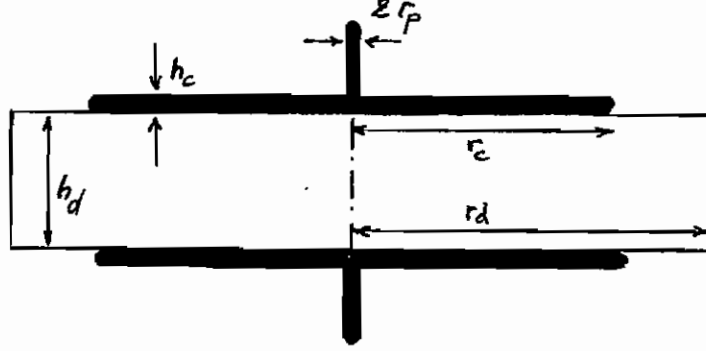


Figure 9: Circular disk capacitor.

3.2 Longitudinal electron motion and interfacial polarisation

The expressions Eqn (1), (2) and (3) for the condenser formed in y -direction remain of course valid, but now with:-

- $\epsilon = \epsilon' - j\epsilon''$: effective complex permittivity of the condenser formed by the DUT and both ohmic contact plates;
- $\tau_c = R_{cl}C_c$: time constant of equivalent circuit for contact plate layers in longitudinal direction;
- $\tau_d = R_dC_d$: time constant of equivalent circuit for DUT;
- $\tau = RC = \frac{R_{cl}R_d(C_c+C_d)}{R_{cl}+R_d} = \frac{R_{cl}\tau_d+R_d\tau_c}{R_{cl}+R_d} = \frac{\epsilon'_c h_d + \epsilon'_d h_c}{\sigma_c h_d + \sigma_d h_c}$: time constant of equivalent circuit for two-layer condenser;
- $R_{cl} = h_c/(\pi r_c^2 \sigma_c)$: equivalent circuit resistance for contact plate layers;
- $C_c = \pi r_c^2 \epsilon'_c / h_c$: equivalent circuit capacitance for contact plate layers;
- $R_d = h_d/(\pi r_c^2 \omega \epsilon''_d)$: equivalent circuit resistance for DUT (note: expression contains r_c , not r_d !);
- $C_d = \pi r_c^2 \epsilon'_d / h_d$: equivalent circuit capacitance for DUT (note: expression contains r_c , not r_d !);
- $C_o = \pi r_c^2 \epsilon_o / (h_d + 2h_c)$: capacitance of a vacuum condenser.

3.3 Transverse electron motion

Eqn (13) remains valid, because the equivalent circuit for the disk capacitor is the same as that for the 1-dimensional capacitor. However, in this case the definitions and equivalent expressions for the various network elements are:-

- $C_d = \frac{\epsilon' \pi (r_c^2 - r_p^2)}{h_d}$: equivalent network capacitance of the DUT;
- $R_d = \frac{h_d}{\omega \epsilon'' \pi (r_c^2 - r_p^2)}$: equivalent network resistance of the DUT;
- $R_{ct}(r - r_p)$: equivalent network resistance of the conducting plate across a distance $r - r_p$ ($r_p \leq r \leq r_c$) in the transverse direction.
- $\epsilon = \epsilon' - j\epsilon''$: complex permittivity of the DUT;
- $\sigma_c = \rho_c^{-1}$: surface conductivity of the conducting plating material;
- r_c, h_c : radius and thickness of conducting plating disk;
- h_d : thickness of the DUT;
- r_p : radius of the (circular cylindrical) probe wire.

To obtain an approximate⁵ expression for R_{ct} , consider first a conducting ring of width dr and radius r . For radially directed currents, such a ring has an incremental resistance:-

$$dR_{ct} = \frac{dr}{2\pi r h_c \sigma_c} \quad (22)$$

across a radial distance dr . Hence for an annular disk of width $r - r_p$:-

$$R_{ct}(r) = \int_{r_p}^r dR_{ct} = \frac{\log(r/r_p)}{2\pi h_c \sigma_c} \quad (23)$$

The average resistance (expectancy value) for the transverse resistance of such a disk is:-

$$\langle R_{ct} \rangle = \frac{\int_0^{2\pi} d\phi \int_{r_p}^{r_c} r R_{ct}(r) dr}{\int_0^{2\pi} d\phi \int_{r_p}^{r_c} r dr} \quad (24)$$

Using partial integration, we calculate:-

$$\int x \log x \, dx = \frac{x^2}{2} \left(\log x - \frac{1}{2} \right) \quad (25)$$

which is used to obtain $\langle R_{ct} \rangle$:-

$$\langle R_{ct} \rangle = (2\pi \sigma_c h_c)^{-1} \frac{\left(\frac{r_c}{r_p}\right)^2 \left[\log\left(\frac{r_c}{r_p}\right) - \frac{1}{2} \right] + \frac{1}{2}}{\left(\frac{r_c}{r_p}\right)^2 - 1} \quad (r_c > r_p) \quad (26)$$

⁵see remark for one-dimensional capacitor

The expectancy value for the overall loss tangent is then:-

$$\begin{aligned}\langle \tan \delta_T \rangle &= \frac{R_d^{-1} + 2\langle R_{ct} \rangle R_d^{-2} [1 + (\omega C_d R_d)^2]}{\omega C_d} \\ &= \frac{\epsilon''}{\epsilon'} + \frac{\pi r_p^2}{h_c h_d} \left\{ \left(\frac{r_c}{r_p} \right)^2 \left[\log \left(\frac{r_c}{r_p} \right) - \frac{1}{2} \right] + \frac{1}{2} \right\} \frac{\omega \epsilon'}{\sigma_c} \left[1 + \left(\frac{\epsilon''}{\epsilon'} \right)^2 \right] \quad (27)\end{aligned}$$

$$\triangleq \tan \delta_d + \omega K_2 C_d \quad (28)$$

with

$$\tan \delta_d = \frac{\epsilon''}{\epsilon'} \quad (29)$$

$$K_2 = (\sigma_c h_c)^{-1} \frac{\left(\frac{r_c}{r_p} \right)^2 \left[\log \left(\frac{r_c}{r_p} \right) - \frac{1}{2} \right] + \frac{1}{2}}{\left(\frac{r_c}{r_p} \right)^2 - 1} [1 + \tan^2 \delta_d] \quad (30)$$

Again, the overall loss tangent consists of two terms: the loss tangent of the DUT and a contribution formed by both the DUT and the ohmic contact plate. For $r_c \gg r_p$, K_2 is approximately proportional to $\log(r_c/r_p)$ which itself shows a gradually weaker increase the further its argument increases. Thus, with $r_c > r_p$, the second term in Eqn (28) shows an increase with (r_c/r_p) faster than $(r_c/r_p)^2$ but slower than $(r_c/r_p)^3$.

3.4 Numerical results

A number of numerical simulations for the above model have been performed. The effective permittivity $\epsilon = \epsilon' - j\epsilon''$ of the condensor is computed first, from input data (estimates) for the individual layers, $\epsilon'_d - j\epsilon''_d$ and $\epsilon'_c - j\sigma_c/(\omega\epsilon_o)$. This ϵ is then used to compute losses due to transverse currents.

Figs 10–15 show the dependency of the overall loss tangent $\tan \delta_T$ as a function of various variables for several constant parameter values:-

- Fig 10: $\tan \delta_T(f)$ for selected values of r_c ;
- Fig 11: $\tan \delta_T(f)$ for selected values of r_p ;
- Fig 12: $\tan \delta_T(r_c)$ for selected values of f ;
- Fig 13: $\tan \delta_T(h_c)$ for selected values of f ;
- Fig 14: $\tan \delta_T(h_c)$ for selected values of r_c ;
- Fig 15: $\tan \delta_T(\tan \delta_d)$ for selected values of f .

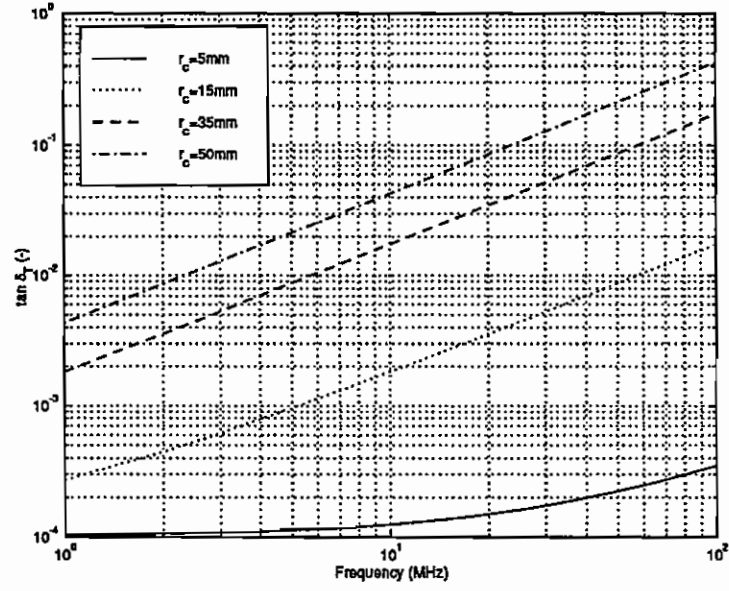


Figure 10: $\tan \delta_T$ as a function of f for $r_c = 5\text{mm}$, 15mm , 35mm and 50mm . $r_p = 4\text{mm}$, $h_c = 0.1\text{ }\mu\text{m}$, $h_d = 1\text{mm}$, $\epsilon'_d/\epsilon_o = 10$, $\tan \delta_d = 10^{-4}$, $\sigma_c = 2.1 \times 10^7\text{ S/m}$.

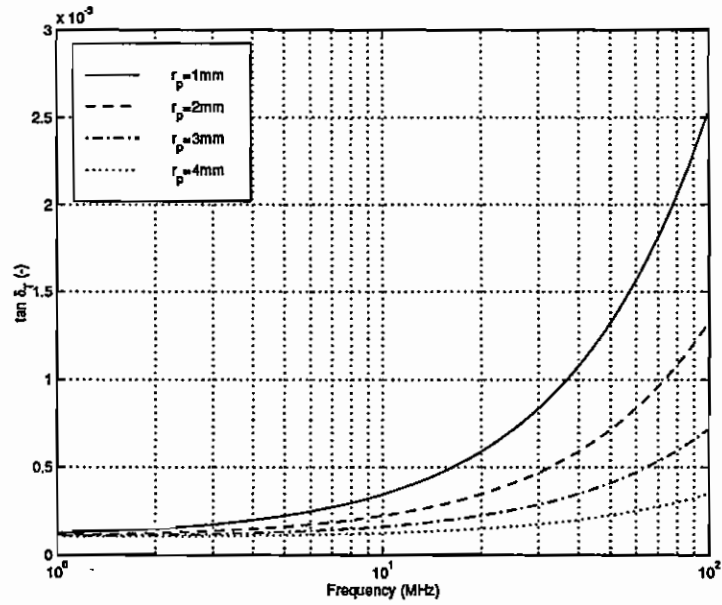


Figure 11: $\tan \delta_T$ as a function of f for $r_p = 1\text{mm}$, 2mm , 3mm and 4mm . $r_c = 5\text{mm}$, $h_c = 0.1\text{ }\mu\text{m}$, $h_d = 1\text{mm}$, $\epsilon'_d/\epsilon_o = 10$, $\tan \delta_d = 10^{-4}$, $\sigma_c = 2.1 \times 10^7\text{ S/m}$.

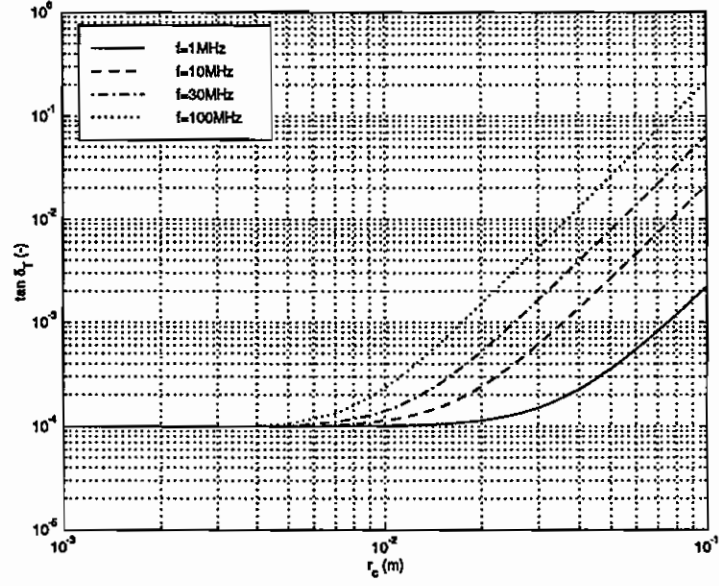


Figure 12: $\tan \delta_T$ as a function of r_c for $f = 1\text{MHz}$, 10MHz , 30MHz and 100MHz . $r_p = 4\text{mm}$, $h_c = 0.1\text{ }\mu\text{m}$, $h_d = 1\text{mm}$, $\epsilon'_d/\epsilon_o = 10$, $\tan \delta_d = 10^{-4}$, $\sigma_c = 2.1 \times 10^7\text{ S/m}$.

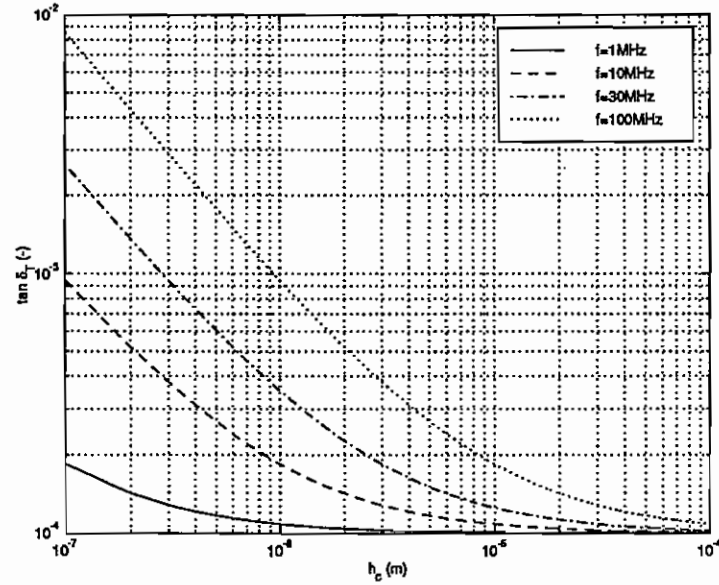


Figure 13: $\tan \delta_T$ as a function of h_c for $f = 1\text{MHz}$, 10MHz , 30MHz and 100MHz . $r_c = 35\text{mm}$, $r_p = 4\text{mm}$, $h_d = 1\text{mm}$, $\epsilon'_d/\epsilon_o = 10$, $\tan \delta_d = 10^{-4}$, $\sigma_c = 2.1 \times 10^7\text{ S/m}$.

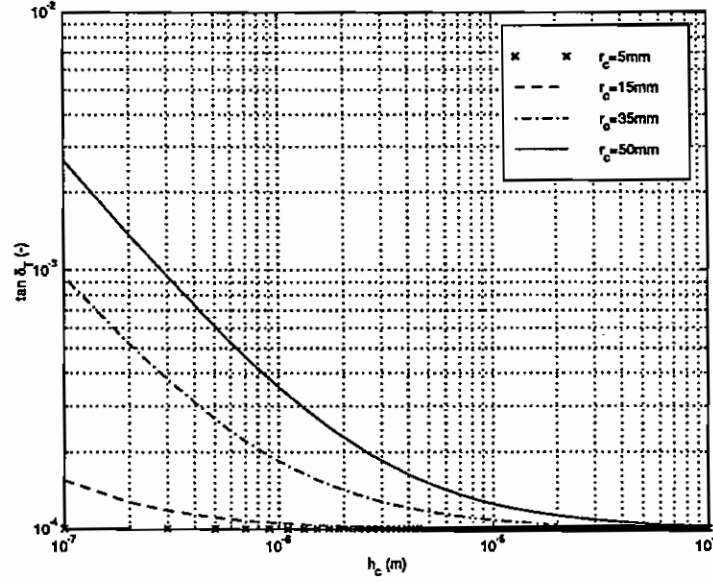


Figure 14: $\tan \delta_T$ as a function of h_c for $r_c = 5\text{mm}$, 15mm , 35mm and 50mm . $f = 30\text{MHz}$, $r_p = 4\text{mm}$, $h_d = 1\text{mm}$, $\epsilon'_d/\epsilon_o = 10$, $\tan \delta_d = 10^{-4}$, $\sigma_c = 2.1 \times 10^7 \text{ S/m}$.

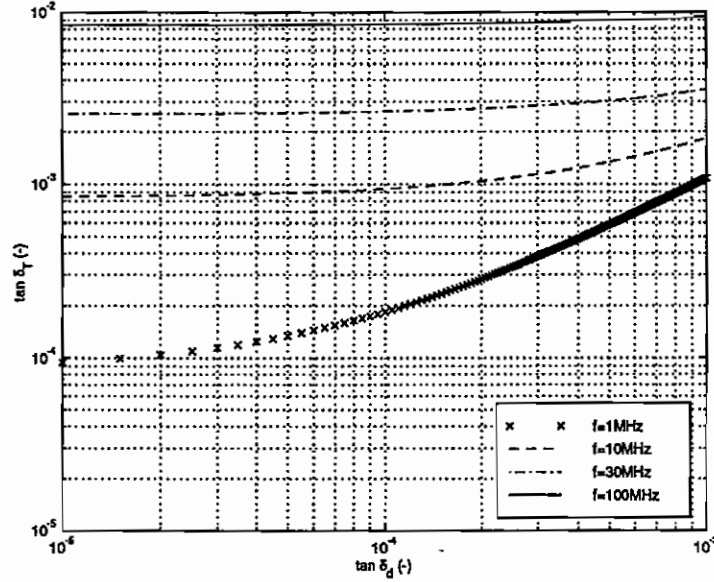


Figure 15: $\tan \delta_T$ as a function of $\tan \delta_d$ for $f = 1\text{MHz}$, 10MHz , 30MHz and 100MHz . $r_c = 35\text{mm}$, $r_p = 4\text{mm}$, $h_c = 0.1\text{mm}$, $h_d = 1\text{mm}$, $\epsilon'_d/\epsilon_o = 10$, $\sigma_c = 2.1 \times 10^7 \text{ S/m}$.

4 Interdigital capacitor

4.1 Introduction

A second commonly used capacitor is the multi-element parallel-plate capacitor with interdigital 'fingers'. Fig 16 shows a cross-section of the configuration of interest.

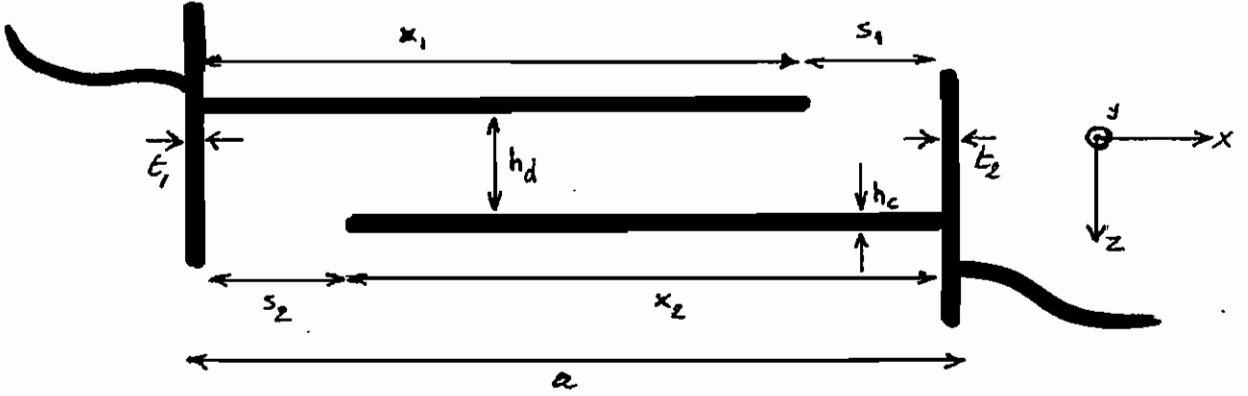


Figure 16: Interdigital plate capacitor (1 element).

The various parameters refer to:-

- x_1, x_2 : lengths in x -direction of the two 'fingers' of each element in the total capacitor;
- s_1, s_2 : spacing in x -direction between the fingers of electrodes 1, 2 and the opposite contact plates of electrodes 2, 1 in one cell;
- t_1, t_2 : thickness of the contact plates of the respective electrodes which support the fingers;
- h_c : thickness, measured in z -direction, of each finger;
- h_d : height of the DUT (distance between fingers in z -direction);
- a : total width of overall structure in x -direction;
- b : total depth of the overall structure in y -direction.

With these definitions, the boundary condition:-

$$a = t_1 + x_1 + s_1 + t_2 = t_2 + x_2 + s_2 + t_1 \quad (31)$$

i.e.:-

$$x_1 + s_1 = x_2 + s_2 \quad (32)$$

applies. The structure shows a periodicity in z -direction, hence only one cell needs to be analysed.

4.2 Longitudinal electron motion and interfacial polarisation

The expressions Eqn (1), (2) and (3) for the Maxwell-Wagner condenser formed in y -direction remain of course valid, but with:-

- $\epsilon = \epsilon' - j\epsilon''$: effective complex permittivity of the condensor formed by the DUT and one pair of ohmic fingers (one cell);
- $\tau_c = R_{cl}C_c$: time constant of equivalent circuit for contacts (in y -direction);
- $\tau_d = R_dC_d$: time constant of equivalent circuit for DUT;
- $\tau = RC = \frac{R_{cl}R_d(C_c+C_d)}{R_{cl}+R_d} = \frac{R_{cl}\tau_d+R_d\tau_c}{R_{cl}+R_d} = \frac{\epsilon'_c h_d + \epsilon'_d h_c}{\sigma_c h_d + \sigma_d h_c}$: time constant of equivalent circuit for two-layer condensor;
- $R_{cl} = h_c/[2(x_1 - s_2)b\sigma_c] \equiv h_c/[2(x_2 - s_1)b\sigma_c]$: equivalent circuit resistance for contact plate fingers. The factor '1/2' originates from taking the average height in the condensor plates, because the source for the longitudinal currents is now placed sideways. This assumes that the current density is uniform throughout the cross-section, which is an approximation;
- $C_c = (x_1 - s_2)b\epsilon'_c/h_c$: equivalent circuit capacitance for contact plate fingers;
- $R_d = h_d/[(x_1 - s_2)b\omega\epsilon''_d]$: resistance of equivalent circuit for contact plate fingers (note: expression contains $x_1 - s_2$, not x_1 !);
- $C_d = (x_1 - s_2)b\epsilon'_d/h_d$: capacitance of equivalent circuit for contact plate fingers (note: expression contains $x_1 - s_2$, not x_1 !);
- $C_o = (x_1 - s_2)b\epsilon_o/(h_d + 2h_c)$: capacitance of a vacuum condensor.

4.3 Transverse electron motion

The expression Eqn (13) can still be used but with different definitions and equivalent expressions for the various network elements:-

- $C_d = \frac{\epsilon'(x_1-s_2)b}{h_d}$: equivalent network capacitance of the DUT;
- $R_d = \frac{h_d}{\omega\epsilon''(x_1-s_2)b}$: equivalent network resistance of the DUT;
- $R_{ct1}(x - s_2) \approx \frac{x_1+s_2}{\sigma_c h_c b} + \frac{x-s_2}{\sigma_c h_c b}$: equivalent network resistance⁶ of finger 1 across a distance $x - s_2$ ($s_2 \leq x \leq x_1$) in the transverse direction. This expression is based on the previously obtained equivalent result for the one-dimensional capacitor;
- $R_{ct2}(x - s_1) \approx \frac{x_2+s_1}{\sigma_c h_c b} + \frac{x-s_1}{\sigma_c h_c b}$: equivalent network resistance of finger 2 across a distance $x - s_1$ ($s_1 \leq x \leq x_2$) in the transverse direction;
- $R_{ct} = R_{ct1} + R_{ct2}$: total two-finger equivalent resistance;

⁶Note earlier remark on approximation of uniform current density for computing R_{ct1} .

- $\epsilon = \epsilon' - j\epsilon''$: complex permittivity of the DUT;
- $\sigma_c = \rho_c^{-1}$: surface conductivity of the conducting plating material;

Integrating yields the expectancy values $\langle R_{ct1} \rangle$ and $\langle R_{ct2} \rangle$:-

$$\langle R_{ct1} \rangle = \frac{t_1 + s_2}{\sigma_c h_c b} + \frac{\int_0^b dy \int_0^{x_1-s_2} R_{ct1}(x) dx}{\int_0^b dy \int_0^{x_1-s_2} dx} = \frac{t_1 + s_2}{\sigma_c h_c b} + \frac{x_1 - s_2}{2\sigma_c h_c b} = \frac{x_1 + 2t_1 + s_2}{2\sigma_c h_c b} \quad (33)$$

$$\langle R_{ct2} \rangle = \frac{t_2 + s_1}{\sigma_c h_c b} + \frac{\int_0^b dy \int_0^{x_2-s_1} R_{ct2}(x) dx}{\int_0^b dy \int_0^{x_2-s_1} dx} = \frac{t_2 + s_1}{\sigma_c h_c b} + \frac{x_2 - s_1}{2\sigma_c h_c b} = \frac{x_2 + 2t_2 + s_1}{2\sigma_c h_c b} \quad (34)$$

Note the different constant factors in each one of these expressions: the factor '1/2' is a result of the spatial averaging of the capacitor's resistance in x -direction. For the part of the finger that does not form a capacitor with the other finger, the multiplying factor is '1' as the current distribution in this section is constant. The expectancy value for the total transverse resistance is obtained, using Eqn (31), as:-

$$\langle R_{ct} \rangle = \langle R_{ct1} \rangle + \langle R_{ct2} \rangle = \frac{a}{\sigma_c h_c b} \quad (35)$$

The expectancy value for the overall loss tangent is therefore:-

$$\begin{aligned} \langle \tan \delta_T \rangle &= \frac{R_d^{-1} + \langle R_{ct} \rangle R_d^{-2} [1 + (\omega C_d R_d)^2]}{\omega C_d} \\ &= \frac{\epsilon''}{\epsilon'} + 2 \frac{a(x_1 - s_2)}{h_c h_d} \frac{\omega \epsilon'}{\sigma_c} \left[1 + \left(\frac{\epsilon''}{\epsilon'} \right)^2 \right] \\ &= \tan \delta_d + \omega K_3 C_d \end{aligned} \quad (36)$$

with

$$\tan \delta_d = \frac{\epsilon''}{\epsilon'} \quad (37)$$

$$K_3 = \frac{2a}{\sigma_c h_c b} [1 + \tan^2 \delta_d] \quad (38)$$

It is interesting to note that K_3 is independent of $x_{1,2}$ and $s_{1,2}$, i.e. the detailed lengths and spacings of the individual fingers inside the capacitor: K_3 only depends on the overall (outer) aspect ratio a/b . Eqn (36) shows that the ohmic contribution of the electrodes to the overall loss tangent of the capacitor is only dependent, linearly, upon the overlap $x_1 - s_2 \equiv x_2 - s_1$. Of course, because of the additional dependence upon a , it is also implicitly dependent upon t_1 and t_2 .

The above analysis and results hold for a single capacitive element (one finger pair). For an interdigital capacitor consisting of N such parallel finger pairs (N parallel capacitors formed

by $N + 1$ fingers), the equivalent circuit admittance is NY and the above expression for the loss tangent remains applicable, because:-

$$\tan \delta_T = \frac{\text{Re}(NY)}{\text{Im}(NY)} = \frac{\text{Re}(Y)}{\text{Im}(Y)} \quad (39)$$

4.4 Total electron motion

Substituting Eqn (3) into Eqn (36-38) the overall loss tangent, containing contributions of dielectric loss and tranverse and longitudinal current resistive losses in the contact plate and fingers, is obtained.

4.5 Numerical results

A number of numerical simulations for the above model have been performed for a symmetric capacitor type, i.e. for which $x_1 = x_2 \triangleq x$, $s_1 = s_2 \triangleq s$, $t_1 = t_2 \triangleq t$. Hence the finger length and finger spacing are related through:-

$$s = a - 2t - x \quad (40)$$

and the finger overlap distance (capacitor width) is:-

$$x - s = 2(x + t) - a \quad (41)$$

The effective permittivity $\epsilon = \epsilon' - j\epsilon''$ of the condenser is computed first, from input data (estimates) for the individual layers, $\epsilon'_d - j\epsilon''_d$ and $\epsilon'_c - j\sigma_c/(\omega\epsilon_o)$. This ϵ is then used to compute losses due to transverse currents.

Figs 17-22 show the dependency of the overall loss tangent $\tan \delta_T$ as a function of various variables for several constant parameter values:-

- Fig 17: $\tan \delta_T(f)$ for selected values of x/a ;
- Fig 18: $\tan \delta_T(f)$ for selected values of h_c/a ;
- Fig 19: $\tan \delta_T(x/a)$ for selected values of f ;
- Fig 20: $\tan \delta_T(h_c/a)$ for selected values of f ;
- Fig 21: $\tan \delta_T(h_c/a)$ for selected values of x/a ;
- Fig 22: $\tan \delta_T(\tan \delta_d)$ for selected values of f .

By virtue of Eqn (36), the dependencies of $\tan \delta_T$ on h_d/a are qualitatively the same as to those on h_c/a .

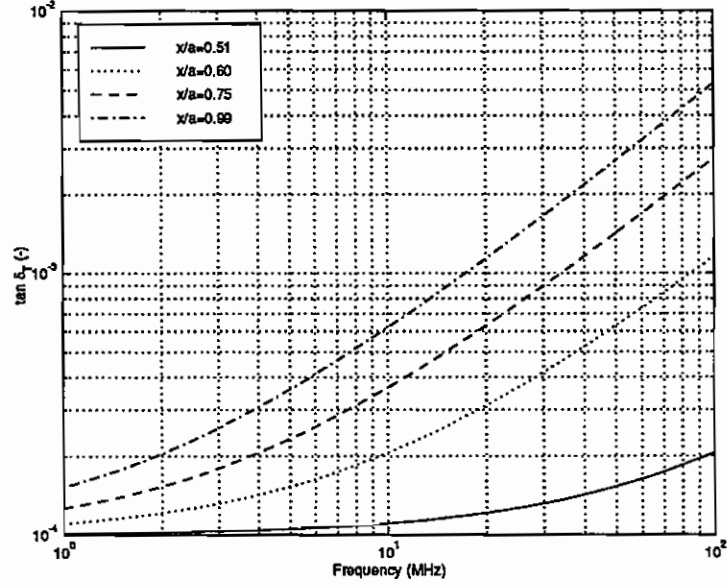


Figure 17: $\tan \delta_T$ as a function of f for $x/a = 0.51, 0.60, 0.75$ and 0.99 . $a = b = 10.0002\text{mm}$, $t = h_c = 0.1\ \mu\text{m}$, $h_d = 1\text{mm}$, $\epsilon'_d/\epsilon_o = 10$, $\tan \delta_d = 10^{-4}$, $\sigma_c = 2.1 \times 10^7\ \text{S/m}$.

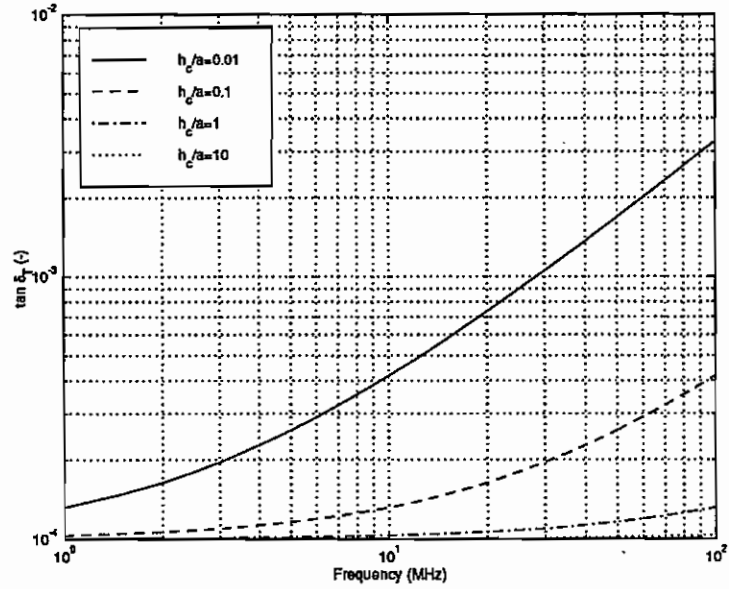


Figure 18: $\tan \delta_T$ as a function of f for $h_c/a = 0.01, 0.1, 1$ and 10 . $a = b = 10.0002\text{mm}$, $x/a = 0.8$, $t = 0.1\ \mu\text{m}$, $h_d = 1\text{mm}$, $\epsilon'_d/\epsilon_o = 10$, $\tan \delta_d = 10^{-4}$, $\sigma_c = 2.1 \times 10^7\ \text{S/m}$.

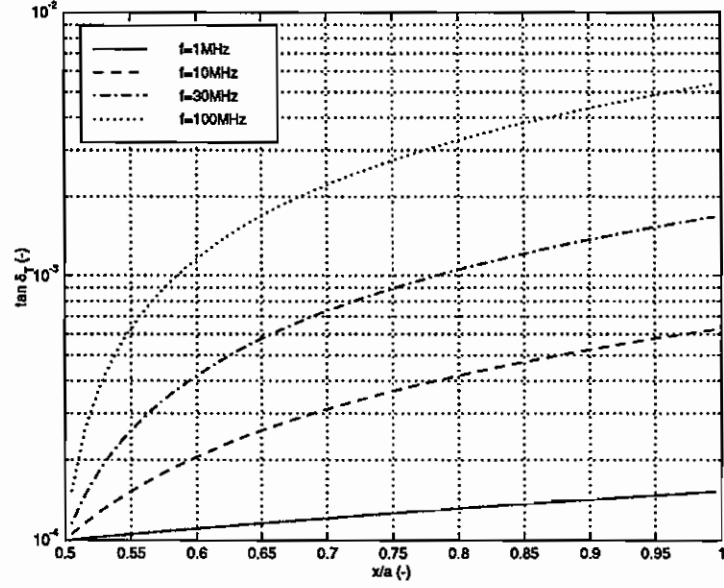


Figure 19: $\tan \delta_T$ as a function of x/a for $f = 1\text{MHz}$, 10MHz , 30MHz and 100MHz . $a = b = 10.0002\text{mm}$, $t = h_c = 0.1 \mu\text{m}$, $h_d = 1\text{mm}$, $\epsilon'_d/\epsilon_o = 10$, $\tan \delta_d = 10^{-4}$, $\sigma_c = 2.1 \times 10^7 \text{ S/m}$.

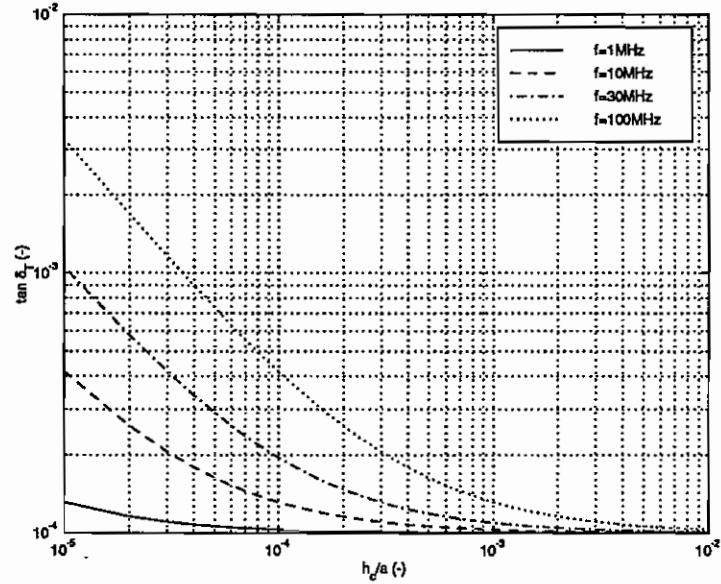


Figure 20: $\tan \delta_T$ as a function of h_c/a for $f = 1\text{MHz}$, 10MHz , 30MHz and 100MHz . $a = b = 10.0002\text{mm}$, $x/a = 0.8$, $h_d = 1\text{mm}$, $\epsilon'_d/\epsilon_o = 10$, $\tan \delta_d = 10^{-4}$, $\sigma_c = 2.1 \times 10^7 \text{ S/m}$.

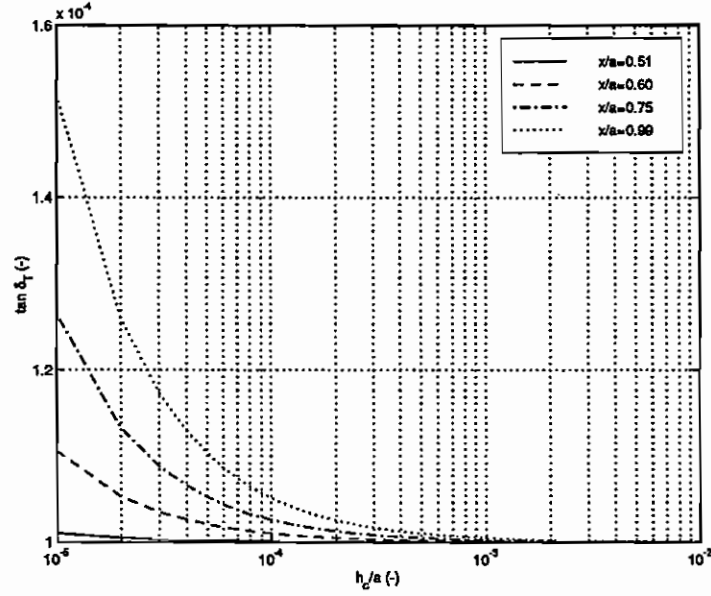


Figure 21: $\tan \delta_T$ as a function of h_c/a for $x/a = 0.51, 0.60, 0.75$ and 0.99 . $f = 1\text{MHz}$, $h_d = 1\text{mm}$, $\epsilon'_d/\epsilon_o = 10$, $\tan \delta_d = 10^{-4}$, $\sigma_c = 2.1 \times 10^7 \text{ S/m}$.

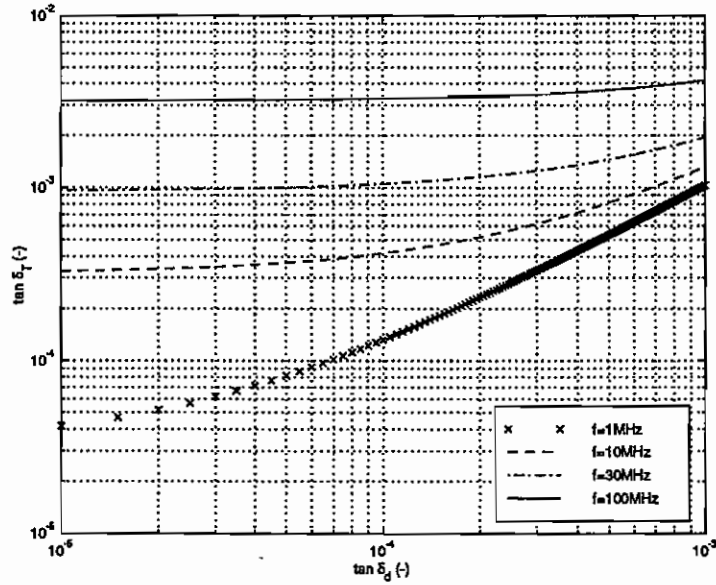


Figure 22: $\tan \delta_T$ as a function of $\tan \delta_d$ for $f = 1\text{MHz}, 10\text{MHz}, 30\text{MHz}$ and 100MHz . $a = b = 10.0002\text{mm}$, $x/a = 0.8$, $h_c = 0.1\text{mm}$, $h_d = 1\text{mm}$, $\epsilon'_d/\epsilon_o = 10$, $\sigma_c = 2.1 \times 10^7 \text{ S/m}$.

5 Proposed experimental verification procedure

To check the accuracy of the proposed model, a sequence of at least five experiments should be performed. We restrict the discussion to the example of a disk capacitor.

First, a DUT is chosen and its $\tan \delta_d$ is estimated. This estimate is required in order to properly select the dimensions and properties of the metalisation, σ_c , h_c , r_c such that its associated loss tangent is of the same order of magnitude of $\tan \delta_d$ in order to obtain a measurable effect. The measured $\tan \delta_T$ for this configuration is then compared with (a) a metalisation that is much thicker or thinner (but same r_c), (b) another metalisation that has a much smaller or larger contact area (but same h_c), (c) another metalisation with the original r_c , h_c but different σ_c , (d) the original metalisation, but on a dielectric with different thickness h_d .

Second, in order to check the validity of the assumptions made on the effect of transverse currents, one of the samples is operated at frequencies of the order of Eqn (19) (this may well be in the microwave region). Alternatively, an extremely wide sample with large contact area ($r_c \sim$ several metres) could be fabricated for use in the original MHz frequency range. Several measurements, close to and half-way in between transverse resonance frequencies should be made. Care must be taken to avoid a longitudinal resonance.

The latter experiment may be very impractical to realize at the frequency range of interest (MHz). A more amenable method may be to excite the DUT under pulsed regime with time gating, after which the response is Fourier transformed. The anticipated result for $\epsilon'(\omega)$ is then the typical duct-shaped response (dielectric relaxation), with a resonance in the optical branch of the duct indicating the transverse resonance. The $\epsilon''(\omega)$ response is expected to show two resonance peaks: one at ω_d and one at the fundamental ω_t .

6 Supporting MATLAB demonstration and CAE software

The MATLAB program:-

Breughel(C:)\MATLAB\ghill\caplowloss.m

and the additional subroutines:-

Breughel(C:)\MATLAB\ghill\epsTp.m
Breughel(C:)\MATLAB\ghill\epsTpp.m

have been developed during this study. They can be used can be used for the inversion procedure of estimating $\epsilon'_d - j\epsilon''_d$ from measured values of $\epsilon'_T - j\epsilon''_T$. Additional information, description and user instructions can be found in the preamble of the main program.

A set of additional MATLAB demonstration programs:-

Breughel(C:)\MATLAB\ghill\caplowlossdemo3.m
 Breughel(C:)\MATLAB\ghill\caplowlossdemo4.m
 Breughel(C:)\MATLAB\ghill\caplowlossdemo5.m

have been developed and used to generate the numerical examples in the previous sections for the one-dimensional capacitor, disk capacitor and interdigital capacitor, respectively. These codes can be used further to easily recompute $\tan \delta_T$ for values of dimensions and material parameters other than those used in the previous examples.

7 Conclusion

The effects of transverse and longitudinal currents in the ohmic contact plates for a DUT have been assessed individually for three types of parallel-plate capacitors: one-dimensional, circular disk and interdigital. The transverse currents were found to constitute a transient effect in the quasi-static approximation. They contribute to the overall loss tangent in a manner indicated by Eqn (16). The longitudinal currents for non-perfectly conducting ohmic contacts cause interfacial polarisation and give rise to an overall loss tangent given by Eqn (4), with its 1st order approximation Eqn (6) for good conductors. For each one of the three types of capacitor investigated, it was found that the overall loss tangent can be written in the form:-

$$\tan \delta_T = \tan \delta_d + \omega K C_d [1 + \tan^2 \delta_d] \quad (42)$$

where C_d is the equivalent circuit capacitance for that type of capacitor and the factor K (in units Ω) takes the following forms (cfr. previous sections for definition of symbols):-

- for the one-dimensional capacitor:-

$$K = \frac{2(x_c - x_p)}{y_c h_c \sigma_c} \quad (43)$$

- for the circular disk capacitor:-

$$K = (\sigma_c h_c)^{-1} \frac{\left(\frac{r_c}{r_p}\right)^2 \left[\log\left(\frac{r_c}{r_p}\right) - \frac{1}{2}\right] + \frac{1}{2}}{\left(\frac{r_c}{r_p}\right)^2 - 1} \quad (44)$$

- for the interdigital capacitor:-

$$K = \frac{2a}{bh_c \sigma_c} \quad (45)$$

The present analysis does not include – but can be readily extended to include – fringing effects, which involves additional correction formulae (e.g. [2, 3]).

References

- [1] von Hippel, A. R.: *Dielectrics and Waves*, Chapman and Hall, 1954, pp 228–229
- [2] Feynman, R.P., Leighton, R. A., Sands, M.: *The Feynman Lectures on Physics*; vol II, Addison-Wesley, Reading, MA, 1966, p 6–12
- [3] Hewlett–Packard Technical Note; *Impedance Measurement Basics*, p 11–28