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Stress transfer mechanics for multiple ply laminates subject to bending

by

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SUMMARY

A methodology based on a new analytical model has been developed for predicting stress transfer due to ply cracking in a cross-ply laminate subject to out-of-plane bending. The model is valid for multiple ply laminates that can be unbalanced (i.e. non-symmetrical). Full account is taken of the combined effects of in-plane uniaxial loading (leading to out-of-plane bending when the laminate is non-symmetrical) and out-of-plane bending. Thermal residual stresses and the associated bending arising from a through-thickness variation of the temperature distribution are taken into account, although ply refinement methods are needed to deal with any through-thickness temperature variation. The representations for the stress and displacement fields satisfy exactly the equilibrium equations and both traction and displacement continuity conditions at the interfaces between plies. The stress-strain equations are satisfied either exactly or in an average sense for each ply or ply layer. Stress transfer is governed by sets of fourth order ordinary differential equations that are solved by standard methods. Solutions of the differential equations are used to determine the stress and displacement distributions at all points in a laminate subject to combined uniaxial loading and out-of-plane bending when equally spaced cracks are present in any of the 90 degree plies.

The stress and displacement distributions are used to predict the effective thermo-elastic constants relevant to the bending of a laminate. Example predictions are given that illustrate the capabilities of the model, including an investigation of the effect of ply refinement techniques on the six effective thermo-elastic constants of a cracked laminate for a range of crack densities.

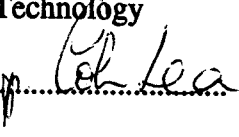
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1. Introduction

Predicting the initiation and progressive formation of ply cracks in laminated composites subject to the complex loading that is expected in composite components is an important technological objective. This goal has recently been achieved [1, 2] for the case of general symmetric laminates that are subject to general in-plane loading. A practically important deformation mode, that has to date been rather neglected in the literature, concerns the prediction of ply crack formation in laminates that are subject to out-of-plane bending. A precursor to the development of such prediction methods is the need for a model to predict the effective thermo-elastic constants for a cracked laminate that is subject to combined uniaxial loading and out-of-plane bending. Such a model will be developed in this report, although a brief description of the work is to be presented elsewhere [3].

Li et al [4] have developed a finite strip analysis of cracked laminates that is capable of handling multiple ply laminates and out-of-plane bending deformation modes. Their technique is based on a series representation for the displacement field combined with the use of minimisation principles applied to the strain energy. Their analysis does not take account of the effects of thermal residual stresses.

The objective of this report is to describe a new model of stress transfer for unbalanced multiple-ply cross-ply laminates subject to out-of-plane bending for which plane strain conditions prevail. Account is taken of the combined effects of axial loading and bending, and of the effects of through-thickness temperature variations that also lead to out-of-plane bending when laminates are symmetric, unsymmetric and/or asymmetrically cracked. The assumption of plane strain means that there is bending only in one direction. Thus the iconoclastic bending associated with thick plates cannot be treated.

The new model is based on a representation for the stress and displacement fields that satisfy exactly the equilibrium equations and both traction and displacement continuity conditions at the interfaces between plies and ply elements if ply refinement techniques are employed. The stress-strain equations are satisfied either exactly or in an average sense for each ply or ply layer. Extensive use is made of recurrence relations that arise by imposing the continuity of tractions and displacements between the layers representing the laminate. Stress transfer is governed by sets of fourth order ordinary differential equations that are solved by standard methods. Solutions of the differential equations are used to determine the stress and displacement distributions at all points in a laminate subject to combined uniaxial loading and out-of-plane bending when equally spaced cracks are present in any of the 90 degree plies.

Because the analysis is developed for laminates that may be non-symmetrical, hybrid in nature so that plies are made of more than one type of material, subject to combined uniaxial loading, out-of-plane bending, and non-uniform through-thickness temperature distribution, the effective stress/strain relations describing the deformation characteristics are complex in nature but predictable. The major difficulty encountered is characterising the various thermo-elastic constants needed to describe behaviour. A variety of effective constants are defined and methods developed for their prediction as a function of laminate geometry and crack density.

2. Geometry and basic equations

While a composite laminate is normally considered to comprise a set of plies perfectly bonded together, it is useful to introduce the concept of perfectly bonded composite layers where there may be more than one layer allocated to each ply of the laminate. The refinement of plies into such layers will enable through-thickness stress, displacement and temperature variations in each ply to be investigated, leading to more accurate modelling of the stress and displacement fields. A laminate consisting of $N+1$ perfectly bonded layers of 0° and 90° plies is considered within a Cartesian coordinate system such that the x -axis is in the through-thickness direction, the y -axis is in the axial direction while the z -axis is in the in-plane transverse direction. The origin is selected to lie on the upper surface of the laminate, i.e. it lies in the first of the layers of the laminate which are labelled $i = 1 \dots N+1$ from top to bottom. The layer interfaces are located on the planes $x = x_i$, $i = 1 \dots N$ where $x_0 = 0$ and $x_{N+1} = h$, the total thickness of the laminate. The thickness of the i^{th} layer is denoted by $h_i = x_i - x_{i-1}$.

The modelling assumes that the laminate composite is in equilibrium such that the following equilibrium equations are satisfied for each layer in the laminate

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0, \quad (1)$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0, \quad (2)$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0, \quad (3)$$

where the stress tensor is symmetric such that

$$\sigma_{xy} = \sigma_{yx}, \quad \sigma_{xz} = \sigma_{zx}, \quad \sigma_{zy} = \sigma_{yz} \quad (4)$$

The displacements within the i^{th} layer in the x , y , and z directions are denoted by u_i , v_i , and w_i respectively. The strains in the i^{th} layer are then given by

$$\epsilon_{xx}^i = \frac{\partial u_i}{\partial x}, \quad \epsilon_{yy}^i = \frac{\partial v_i}{\partial y}, \quad \epsilon_{zz}^i = \frac{\partial w_i}{\partial z}, \quad (5)$$

$$\epsilon_{xy}^i = \epsilon_{yx}^i = \frac{1}{2} \left(\frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial x} \right), \quad \epsilon_{zy}^i = \epsilon_{yz}^i = \frac{1}{2} \left(\frac{\partial v_i}{\partial z} + \frac{\partial w_i}{\partial y} \right), \quad \epsilon_{xz}^i = \epsilon_{zx}^i = \frac{1}{2} \left(\frac{\partial w_i}{\partial x} + \frac{\partial u_i}{\partial z} \right) \quad (6)$$

The stress strain relations for the i^{th} layer are assumed to be of the form

$$\epsilon_{xx}^i = \frac{\sigma_{xx}^i}{E_t^i} - \frac{\nu_a^i}{E_A^i} \sigma_{yy}^i - \frac{\nu_t^i}{E_T^i} \sigma_{zz}^i + \alpha_t^i \Delta T^i, \quad (7)$$

$$\epsilon_{yy}^i = -\frac{\nu_a^i}{E_A^i} \sigma_{xx}^i + \frac{\sigma_{yy}^i}{E_A^i} - \frac{\nu_A^i}{E_A^i} \sigma_{zz}^i + \alpha_A^i \Delta T^i,$$

$$\epsilon_{zz}^i = -\frac{\nu_t^i}{E_T^i} \sigma_{xx}^i - \frac{\nu_A^i}{E_A^i} \sigma_{yy}^i + \frac{\sigma_{zz}^i}{E_T^i} + \alpha_T^i \Delta T^i$$

$$\epsilon_{xy}^i = \frac{\sigma_{xy}^i}{2\mu_a^i}, \quad \epsilon_{xz}^i = \frac{\sigma_{xz}^i}{2\mu_t^i}, \quad \epsilon_{yz}^i = \frac{\sigma_{yz}^i}{2\mu_A^i}$$

The parameters E^i , ν^i , μ^i and α^i denote the Young's modulus, Poisson's ratio, shear modulus and thermal expansion coefficient in the i^{th} layer of the laminate. The subscripts 'A' and 'a' refer respectively to axial thermoelastic constants involving in-plane and out-of-plane stresses and deformations. The subscripts T and t refer respectively to transverse constants for in-plane and out-of-plane stresses and deformations. The parameter ΔT^i is the difference between the current temperature of the i^{th} layer (assumed uniform in each layer so that non-uniform through-thickness temperature distributions, but independent of y and z , can be dealt with by ply refinement techniques where an individual 0° or 90° ply is represented by a number of layers having the same properties) and the stress-free temperature of the laminate.

3. Plane strain

Consider that the bending of the laminate occurs only in the x - y plane such that $w = 0$ for all layers in the laminate. It is also assumed that the u_i and v_i displacements depend only on x and y so that each layer has the same zero strain in the transverse z direction. From these assumptions it follows from (5) and (6) that

$$\epsilon_{zz}^i = 0, \quad \epsilon_{yz}^i = 0, \quad \epsilon_{xz}^i = 0.$$

Now (10) and (11) imply that

$$\sigma_{xz}^i = 0, \quad \sigma_{yz}^i = 0,$$

while (9) and (11) imply that

$$\sigma_{zz}^i = \nu_t^i \sigma_{xx}^i + \nu_A^i \frac{E_T^i}{E_A^i} \sigma_{yy}^i - E_T^i \alpha_T^i \Delta T^i$$

so that σ_{zz}^i is non-zero but independent of z .

On eliminating σ_{zz}^i in (7) and (8) using (13) it can be shown that

$$\epsilon_{xx}^i = \frac{1}{\tilde{E}_t^i} \sigma_{xx}^i - \frac{\bar{v}_a^i}{\tilde{E}_A^i} \sigma_{yy}^i + \alpha_t^i \Delta T^i, \quad (14)$$

$$\epsilon_{yy}^i = -\frac{\bar{v}_a^i}{\tilde{E}_A^i} \sigma_{xx}^i + \frac{1}{\tilde{E}_A^i} \sigma_{yy}^i + \alpha_A^i \Delta T^i, \quad (15)$$

where

$$\frac{1}{\tilde{E}_t^i} = \frac{1}{E_t^i} - \frac{(v_t^i)^2}{E_t^i}, \quad \frac{\bar{v}_a^i}{\tilde{E}_A^i} = \frac{v_a^i + v_A^i v_t^i}{E_A^i}, \quad (16)$$

$$\frac{1}{\tilde{E}_A^i} = \frac{1}{E_A^i} \left[1 - (v_A^i)^2 \frac{E_t^i}{E_A^i} \right] \quad (17)$$

and

$$\alpha_t^i = \alpha_t^i + v_t^i \alpha_T^i, \quad \alpha_A^i = \alpha_A^i + v_A^i \frac{E_t^i}{E_A^i} \alpha_T^i. \quad (18)$$

Thus when considering the plane strain bending of a composite laminate, the following stress-strain relations and equilibrium equations should be satisfied as closely as possible :

$$\epsilon_{xx}^i = \frac{\partial u_i}{\partial x} = \frac{\sigma_{xx}^i}{\tilde{E}_t^i} - \frac{\bar{v}_a^i}{\tilde{E}_A^i} \sigma_{yy}^i + \alpha_t^i \Delta T^i \quad (19)$$

$$\epsilon_{yy}^i = \frac{\partial v_i}{\partial y} = -\frac{\bar{v}_a^i}{\tilde{E}_A^i} \sigma_{xx}^i + \frac{\sigma_{yy}^i}{\tilde{E}_A^i} + \alpha_A^i \Delta T^i, \quad (20)$$

$$\epsilon_{xy}^i = \frac{1}{2} \left(\frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial x} \right) = \frac{\sigma_{xy}^i}{2\mu_a^i}, \quad (21)$$

$$\frac{\partial \sigma_{xx}^i}{\partial x} + \frac{\partial \sigma_{xy}^i}{\partial y} = 0 ,$$

$$\frac{\partial \sigma_{xy}^i}{\partial x} + \frac{\partial \sigma_{yy}^i}{\partial y} = 0 .$$

4. Interfacial and boundary conditions

The origin for coordinates is selected to be midway between two transverse cracks. As the layers are assumed to be perfectly bonded the displacements and interfacial tractions must be continuous through the laminate such that

$$\left. \begin{aligned} \sigma_{xx}^i(x_i, y) &= \sigma_{xx}^{i+1}(x_i, y) \\ \sigma_{xy}^i(x_i, y) &= \sigma_{xy}^{i+1}(x_i, y) \\ u_i(x_i, y) &= u_{i+1}(x_i, y) \\ v_i(x_i, y) &= v_{i+1}(x_i, y) \end{aligned} \right\} \quad \text{for } i = 1 \dots N , \quad |y| \leq L .$$

Symmetry about the plane $y = 0$ implies that

$$\sigma_{xy}(x, 0) = 0 , \quad v(x, 0) = 0 , \quad \text{for } 0 \leq x \leq h ,$$

and the outer surfaces of the laminate are assumed to be stress free so that for $|y| \leq L$

$$\sigma_{xx}^1(0, y) = 0 , \quad \sigma_{xx}^{N+1}(x_{N+1}, y) = 0 , \quad \sigma_{xy}^1(0, y) = 0 , \quad \sigma_{xy}^{N+1}(x_{N+1}, y) = 0$$

Because the surface of the transverse cracks are stress-free

$$\sigma_{yy}^i(x, \pm L) = 0 , \quad \sigma_{xy}^i(x, \pm L) = 0 \quad \text{for } x_{i-1} \leq x \leq x_i ,$$

for cracked layers only. For a uniform distribution of transverse cracks symmetry requires that for uncracked layers

$$\sigma_{xy}^i(x, \pm L) = 0 , \quad \text{for } x_{i-1} \leq x \leq x_i .$$

The axial displacement for all of the uncracked layers is assumed to have the form

$$v_i(x, \pm L) = \pm \left[\varepsilon_c + \hat{\varepsilon}_c x \right] L ,$$

where ε_c and $\hat{\varepsilon}_c$ are parameters whose values characterise the effective axial strain and bending of the cracked laminate. The corresponding values for an uncracked laminate are denoted by ε and $\hat{\varepsilon}$.

5. Representation for stress and displacement fields

5.1 The stress field

When a multiple ply laminate is subject to combined axial loading and plane strain bending, the stress component σ_{yy} in each layer is assumed to be a linear function of x . It then follows from (23) that

$$\sigma_{xy}(x,y) = A(y)x^2 + B(y)x + Q(y) , \quad (30)$$

where $Q(y)$ and $A(y)$ and $B(y)$ are functions of y . Without loss of generality these functions can be redefined, for $x_{i-1} \leq x \leq x_i$, $i=1...N+1$, such that

$$\sigma_{xy}^i(x, y) = A_i'(y) + B_i'(y)\xi_i + Q_i'(y)\xi_i^2 , \quad \text{where } \xi_i = \frac{x - x_{i-1}}{x_i - x_{i-1}} \quad (31)$$

Hence on $x = x_{i-1}$

$$\sigma_{xy}^i(x_{i-1}, y) = A_i'(y) , \quad (32)$$

and on $x = x_i$

$$\sigma_{xy}^i(x_i, y) = A_i'(y) + B_i'(y) + Q_i'(y) . \quad (33)$$

A new function is introduced, $P_i(y)$, where $P_i'(y)$ is the value of this shear stress at the i^{th} interface. The relation (33) may then be rewritten in the form

$$\sigma_{xy}^i(x, y) = P_{i-1}'(y) + [P_i'(y) - P_{i-1}'(y)]\xi_i + Q_i'(y)[\xi_i^2 - \xi_i] \quad i = 1...N+1 . \quad (34)$$

Using this representation, (24)₂ is automatically satisfied such that

$$\sigma_{xy}^i(x_i, y) = \sigma_{xy}^{i+1}(x_i, y) = P_i'(y) , \quad i = 1...N . \quad (35)$$

From (26)_{3A} and (34) it is clear that

$$P_0'(y) = P_{N+1}'(y) \equiv 0 . \quad (36)$$

As these functions are zero for the undamaged laminate, as discussed later, on integrating (36)

$$P_0(y) = P_{N+1}(y) \equiv 0 . \quad (37)$$

On substituting (34) into (22) it can be shown that

$$\sigma_{xx}^i(x, y) = \left[-P_{i-1}''(y) \xi_i + (P_{i-1}''(y) - P_i''(y)) \frac{1}{2} \xi_i^2 - Q_i''(y) \left(\frac{1}{3} \xi_i^3 - \frac{1}{2} \xi_i^2 \right) \right] h_i + S_{i-1}(y), \quad i = 1 \dots N+1 \quad (38)$$

where $S_{i-1}(y)$ is the through-the-thickness stress on $x = x_{i-1}$. By substitution of (34) into (23) it can be shown that

$$\sigma_{yy}^i(x, y) = \frac{1}{h_i} [P_{i-1}(y) - P_i(y)] - \frac{Q_i(y)}{h_i} [2\xi_i - 1] + \sigma_i(x), \quad i = 1 \dots N+1.$$

The quantity $\sigma_i(x)$ is the longitudinal stress experienced by the i^{th} layer of an undamaged laminate when $P_i(y)$ and $Q_i(y)$ are equal to zero for all i .

5.2 The displacement field

Substitution of (38) and (39) into (19) followed by an integration with respect to x leads to

$$\begin{aligned} u_i(x, y) = & \frac{(x - x_{i-1})^2}{12\bar{E}_t^i} \left[2(P_{i-1}''(y) - P_i''(y)) \xi_i - 6P_{i-1}''(y) - Q_i''(y) (\xi_i^2 - 2\xi_i) \right] \\ & - \frac{\bar{v}_a^i}{\bar{E}_A^i} \left[(P_{i-1}(y) - P_i(y)) \xi_i - Q_i(y) (\xi_i^2 - \xi_i) + \sigma_i^*(x) \right] \\ & + (x - x_{i-1}) \alpha_t^i \Delta T^i + \frac{S_{i-1}(y)}{\bar{E}_t^i} + U_{i-1}(y), \quad i = 1 \dots N+1, \end{aligned}$$

where $U_{i-1}(y)$ represents the through-thickness displacement on $x = x_{i-1}$, and where

$$\sigma_i^*(x) = \int_{x_{i-1}}^x \sigma_i(\eta) d\eta, \quad i = 1 \dots N+1.$$

On substituting (34) and (40) into (21) it can be shown that

$$\begin{aligned}
v_i(x, y) = & - \frac{(x - x_{i-1})^3}{120 \tilde{E}_t^i} \left[5(P_{i-1}'''(y) - P_i'''(y)) \xi_i - 20 P_{i-1}'''(y) - Q_i'''(y) (2 \xi_i^2 - 5 \xi_i) \right] \\
& + (x - x_{i-1}) \left[\frac{P_{i-1}'(y)}{\mu_a^i} + \left(\frac{1}{\mu_a^i} - \frac{\tilde{\nu}_a^i}{\tilde{E}_A^i} \right) \left\{ (P_i'(y) - P_{i-1}'(y)) \frac{1}{2} \xi_i + Q_i'(y) \left(\frac{1}{3} \xi_i^2 - \frac{1}{2} \xi_i \right) \right\} \right] \\
& - \frac{S_{i-1}'(y)}{2 \tilde{E}_t^i} (x - x_{i-1})^2 - U_{i-1}'(y) (x - x_{i-1}) + V_{i-1}(y), \quad i = 1 \dots N+1,
\end{aligned} \tag{42}$$

where $V_{i-1}(y)$ is the longitudinal displacement on $x = x_{i-1}$.

5.3 The recurrence relations

By applying the relevant interfacial conditions (24) the following recurrence relationships can be derived, for $i = 1 \dots N$:

$$S_i(y) = \frac{h_i}{6} [Q_i''(y) - 3P_{i-1}'' - 3P_i''(y)] + S_{i-1}(y), \tag{43}$$

$$\begin{aligned}
U_i(y) = & \frac{h_i^2}{12 \tilde{E}_t^i} [Q_i''(y) - 2P_{i-1}''(y) - 4P_i''(y)] \\
& - \frac{\tilde{\nu}_a^i}{\tilde{E}_A^i} [P_{i-1}(y) - P_i(y) + \sigma_i^*(x_i)] + \left(\alpha_t^i \Delta T^i + \frac{S_{i-1}(y)}{\tilde{E}_t^i} \right) h_i + U_{i-1}(y),
\end{aligned} \tag{44}$$

$$\begin{aligned}
V_i(y) = & - \frac{h_i^3}{120 \tilde{E}_t^i} [3Q_i'''(y) - 5P_{i-1}'''(y) - 15P_i'''(y)] \\
& + \frac{h_i}{6} \left[\frac{1}{\mu_a^i} (3P_{i-1}'(y) + 3P_i'(y) - Q_i'(y)) + \frac{\tilde{\nu}_a^i}{\tilde{E}_A^i} (3P_{i-1}'(y) - 3P_i'(y) + Q_i'(y)) \right] \\
& - \frac{h_i^2}{2 \tilde{E}_t^i} S_{i-1}'(y) - h_i U_{i-1}'(y) + V_{i-1}(y).
\end{aligned} \tag{45}$$

6. The undamaged laminate

The laminate is described as undamaged when no cracks have formed and all the stress tensor components are zero with the exception of the σ_{yy} and σ_{zz} as a result of plane strain bending. Before the laminate becomes cracked it behaves in the manner predicted by beam mechanics.

6.1 The stress field

When modelling the behaviour of the undamaged laminate the following equalities are imposed:

$$P_i(y) \equiv 0, \quad \text{for } i = 0 \dots N+1$$

$$Q_i(y) \equiv 0, \quad \text{for } i = 1 \dots N+1$$

$$S_{i-1}(y) \equiv 0, \quad \text{for } i = 1 \dots N$$

with the result that, from (38) and (39)

$$\sigma_{xx}^i \equiv 0, \quad \sigma_{yy}^i = \sigma_i(x), \quad \text{for } i = 1 \dots N+1$$

Denoting the longitudinal stresses experienced by the i^{th} layer to be σ_i and assuming the axial strain to be linear in x the stress-strain relation (20) reduces to the form

$$\varepsilon + \varepsilon x = \frac{\partial v_i}{\partial y} = \frac{\sigma_i(x)}{\tilde{E}_A^i} + \alpha_A^i \Delta T^i. \quad (50)$$

It then follows that for $x_{i-1} \leq x \leq x_i$, $i = 1 \dots N+1$

$$\sigma_i(x) = \tilde{E}_A^i \left[\varepsilon + \varepsilon x - \alpha_A^i \Delta T^i \right],$$

and on integrating with respect to x , it follows from (41) that

$$\sigma_i^*(x) = \tilde{E}_A^i \left[\frac{1}{2} \varepsilon (x^2 - x_{i-1}^2) + (\varepsilon - \alpha_A^i \Delta T^i)(x - x_{i-1}) \right]$$

It is assumed that stresses within the laminate can arise from any combination of mismatched thermal expansion coefficients, an applied in-plane loading arising from an applied axial force F acting at the mid-point between the upper and lower surfaces of the laminate, and an applied bending moment per unit width of laminate M . From mechanical equilibrium

$$F = h \sigma = \sum_{i=1}^{N+1} \int_{x_{i-1}}^{x_i} \sigma_i(x) dx,$$

and

$$M = \sum_{i=1}^{N+1} \int_{x_{i-1}}^{x_i} \left(x - \frac{h}{2} \right) \sigma_i(x) dx \quad (54)$$

On substituting (51) into (53) and (54) the following expressions are obtained

$$h\sigma = \sum_{i=1}^{N+1} \tilde{E}_A^i \left[\frac{1}{2} \tilde{\epsilon} (x_i^2 - x_{i-1}^2) + (\epsilon - \alpha_A^i \Delta T^i) (x_i - x_{i-1}) \right] \quad (55)$$

and

$$M + \frac{1}{2} h^2 \sigma = \sum_{i=1}^{N+1} \tilde{E}_A^i \left[\frac{1}{3} \tilde{\epsilon} (x_i^3 - x_{i-1}^3) + \frac{1}{2} (\epsilon - \alpha_A^i \Delta T^i) (x_i^2 - x_{i-1}^2) \right] \quad (56)$$

It can now be observed that for a laminate with a specified non-uniform temperature distribution, defined by values ΔT^i , subject to a given bending moment per width M , and to an effective applied stress σ , the relations (55) and (56) enable the parameters ϵ and $\tilde{\epsilon}$ to be calculated, and substitution into (51) enables the longitudinal stress distribution in each layer to be found.

6.2 The displacement field

On using (46-49), which apply only for uncracked laminates, and (40) the through-thickness displacement of the undamaged laminate is given by

$$u_i(x, y) = U_{i-1}^0(y) - \frac{\tilde{V}_A^i}{\tilde{E}_A^i} \sigma_i^*(x) + \alpha_A^i \Delta T^i (x - x_{i-1}), \quad i = 1 \dots N+1, \quad (57)$$

where $U_i^0(y)$ is the value of $U_i(y)$ when the laminate is uncracked. The recurrence relation (44) reduces to

$$U_i^0(y) = U_{i-1}^0(y) - \frac{\tilde{V}_A^i}{\tilde{E}_A^i} \sigma_i^*(x_i) + h_i \alpha_A^i \Delta T^i, \quad i = 1 \dots N. \quad (58)$$

The through-thickness displacement of the top surface of the undamaged laminate, at $x = 0$, can be defined in terms of a length R_0 , the radius of curvature of this surface after loading. The exact relationship is given by the well known formula

$$\frac{1}{R_0} = - \frac{\frac{d^2 u}{dy^2}}{\left[1 + \left(\frac{du}{dy} \right)^2 \right]^{3/2}} \quad (59)$$

but for small deflections this reduces to

$$\frac{1}{R_0} = - \frac{d^2 u}{dy^2} . \quad (60)$$

Hence, on defining the through-thickness deflection so that it is zero at $y = 0$ and symmetric about $y = 0$,

$$u_i(0, y) = U_i^0(y) = - \frac{y^2}{2R_0}$$

On using (61) in conjunction with (58) the functions $U_i^0(y)$, $i = 1 \dots N$, may be calculated and substituted in (57) to obtain the displacement components u_i for an uncracked laminate.

From (50) the following expression for the longitudinal displacement v_i of the undamaged laminate is obtained

$$v_i(x, y) = (\varepsilon + \varepsilon x) y .$$

Hence at the interfaces between layers the axial displacements are given by

$$V_i(y) = (\varepsilon + \varepsilon x_i) y \equiv V_i^0(y) , \quad i = 1 \dots N .$$

The relations (57), (58), (61) and (62) enable the displacement components u_i and v_i to be found provided that R_0 is specified. This variable is chosen such that equation (21) is satisfied when the laminate is undamaged with the shear stresses being zero. Thus, it is essential that

$$R_0 = \frac{1}{\varepsilon}$$

7. Derivation of differential equations

This study of crack formation within a laminate centres around the assumption that the composite behaves in a manner that obeys the basic stress strain and temperature relations. It is now necessary to ensure that the functions $P_i(y)$, $Q_i(y)$ are calculated to satisfy this assumption as closely as possible. The axial stress-strain relation (20) cannot be satisfied exactly, for all x , by the representations that have been generated. The approach taken here is to take the average and moment of (20) and demand that the resulting relations, which are then independent of x , are satisfied exactly.

7.1 Averaging

The average of a variable $f(x, y)$ over the interval $x_{i-1} \leq x \leq x_i$ of the i^{th} element is defined to be

$$\bar{f}_i(y) = \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f_i(x, y) dx .$$

On taking the average of (20)

$$\frac{d\bar{v}_i}{dy} = -\frac{\bar{v}_a^i}{\bar{E}_A^i} \bar{\sigma}_{xx}^i + \frac{1}{\bar{E}_A^i} \bar{\sigma}_{yy}^i + \bar{\alpha}_A^i \bar{\Delta T}^i, \quad i = 1 \dots N+1. \quad (66)$$

It can be shown from (38), (39) and (42) that for $i = 1 \dots N+1$

$$\bar{\sigma}_{xx}^i = \frac{h_i}{12} [Q_i''(y) - 4P_{i-1}''(y) - 2P_i''(y)] + S_{i-1}(y), \quad (67)$$

$$\bar{\sigma}_{yy}^i(y) = \frac{1}{h_i} [P_{i-1}(y) - P_i(y) + \sigma_i^*(x_i)], \quad (68)$$

$$\begin{aligned} \bar{v}_i(y) = & -\frac{h_i^3}{360\bar{E}_t^i} [2Q_i'''(y) - 12P_{i-1}'''(y) - 3P_i'''(y)] \\ & + \left[\frac{1}{12\mu_a^i} \{4P_{i-1}'(y) + 2P_i'(y) - Q_i'(y)\} + \frac{\bar{v}_a^i}{12\bar{E}_A^i} \{2P_{i-1}'(y) - 2P_i'(y) + Q_i'(y)\} \right] h_i \\ & - \frac{h_i^2}{6\bar{E}_t^i} S_{i-1}'(y) - \frac{h_i}{2} U_{i-1}'(y) + V_{i-1}(y), \end{aligned} \quad (69)$$

and

$$\bar{\Delta T}^i = \Delta T^i. \quad (70)$$

On substituting (67-70) into (66), it can be shown that for $i = 1 \dots N+1$

$$\begin{aligned}
& \frac{h_i^3}{360 \tilde{E}_t^i} [12P_{i-1}^{''''}(y) + 3P_i^{''''}(y) - 2Q_i^{''''}(y)] + \frac{P_{i-1}''(y)}{6} \frac{2}{\mu_a^i} - \frac{\nu_a^i}{\tilde{E}_A^i} \\
& + \frac{P_i''(y)}{6} \left(\frac{1}{\mu_a^i} - \frac{2\nu_a^i}{\tilde{E}_A^i} - \frac{Q_i''(y)}{12} \left(\frac{1}{\mu_a^i} - \frac{2\nu_a^i}{\tilde{E}_A^i} \right) \right] h_i + \frac{1}{h_i \tilde{E}_A^i} [P_i(y) - P_{i-1}(y)] \\
& + V_{i-1}'(y) + \frac{\nu_a^i}{\tilde{E}_A^i} S_{i-1}(y) - \frac{h_i}{2} U_{i-1}''(y) - \frac{h_i^2}{6 \tilde{E}_t^i} S_{i-1}''(y) - \epsilon - \left(x_i - \frac{h_i}{2} \right) \epsilon = 0
\end{aligned}$$

7.2 The integrated moments

When the laminate is deformed the stress and displacement field should satisfy the integrated moment of the axial stress-strain relation (20), where for any variable $m(x, y)$ its integrated moment for the i^{th} element is defined as

$$\tilde{m}(y) = \int_{x_{i-1}}^{x_i} x m(x, y) dx .$$

On taking moments (20) may be written

$$\frac{d\check{\nu}_i}{dy} = - \frac{\nu_a^i}{\tilde{E}_A^i} \check{\sigma}_{xx}^i + \frac{1}{\tilde{E}_A^i} \check{\sigma}_{yy}^i + \alpha_A^i \Delta \check{T}^i , \quad i = 1 \dots N+1 .$$

A linear combination of the averaged strain equation and the integrated moment equation is defined as

$$\hat{m}_i = \frac{1}{h_i} \tilde{m}_i - x_{i-1} \bar{m}_i ,$$

usefully expressed as

$$\hat{m}_i(y) = \int_{x_{i-1}}^{x_i} \xi_i m_i(x, y) dx ,$$

where ξ_i is defined by (31)₂. When this newly defined function is applied to the stress-strain equation (20), the stress and displacement fields should satisfy the equations

$$\frac{d\hat{\nu}_i}{dy} = - \frac{\nu_a^i}{\tilde{E}_A^i} \hat{\sigma}_{xx}^i + \frac{1}{\tilde{E}_A^i} \hat{\sigma}_{yy}^i + \alpha_A^i \Delta \hat{T}^i , \quad i = 1 \dots N+1 ,$$

that ensures that (73) is satisfied. After the integration of the appropriate terms the following expressions are found for $i = 1 \dots N+1$

$$\sigma_{xx}^i(y) = \frac{h_i^2}{120} [7Q_i''(y) - 25P_{i-1}''(y) - 15P_i''(y)] + \frac{h_i}{2} S_{i-1}(y) , \quad (77)$$

$$\sigma_{yy}^i(y) = \frac{1}{2} [P_{i-1}(y) - P_i(y)] - \frac{1}{6} Q_i(y) + \frac{h_i \tilde{E}_A^i}{6} \varepsilon (3x_i - h_i) + \frac{h_i \tilde{E}_A^i}{2} (\varepsilon - \alpha_A^i \Delta T^i) , \quad (78)$$

$$\begin{aligned} \varphi_i(y) = & - \frac{h_i^4}{5040 \tilde{E}_t^i} [23Q_i'''(y) - 133P_{i-1}'''(y) - 35P_i'''(y)] \\ & + \frac{1}{120 \mu_a^i} (25P_{i-1}'(y) + 15P_i'(y) - 7Q_i'(y)) + \frac{\varphi_a^i}{120 \tilde{E}_A^i} (15P_{i-1}'(y) - 15P_i'(y) + 7Q_i'(y)) \Big] h_i^2 \\ & - \frac{h_i^3}{8 \tilde{E}_t^i} S_{i-1}'(y) - \frac{h_i^2}{3} U_{i-1}'(y) + \frac{h_i}{2} V_{i-1}(y) , \end{aligned} \quad (79)$$

and

$$\Delta \hat{T}^i = \frac{h_i}{2} \Delta T^i . \quad (80)$$

The substitution of (77-80) into (76) and a multiplication by $2/h_i$ lead to, for $i = 1 \dots N+1$

$$\begin{aligned} & \frac{h_i^3}{2520 \tilde{E}_t^i} [133P_{i-1}''''(y) + 35P_i''''(y) - 23Q_i''''(y)] + \left[\frac{P_{i-1}''(y)}{12} \frac{5}{\mu_a^i} - \frac{2\varphi_a^i}{\tilde{E}_A^i} \right. \\ & \quad \left. + \frac{P_i''(y)}{4} \frac{1}{\mu_a^i} - \frac{2\varphi_a^i}{\tilde{E}_A^i} \frac{7Q_i''(y)}{60} \left(\frac{1}{\mu_a^i} - \frac{2\varphi_a^i}{\tilde{E}_A^i} \right) \right] h_i \\ & \quad - \frac{1}{3h_i \tilde{E}_A^i} [3P_{i-1}(y) - 3P_i(y) - Q_i(y)] + V_{i-1}'(y) + \frac{\varphi_a^i}{\tilde{E}_A^i} S_{i-1}(y) \\ & \quad - \frac{2h_i}{3} U_{i-1}''(y) - \frac{h_i^2}{4 \tilde{E}_t^i} S_{i-1}''(y) - \varepsilon - \left(x_i - \frac{h_i}{3} \right) \varepsilon = 0 . \end{aligned} \quad (81)$$

7.3 Expressions for the interfacial displacements

Treating (71) and (81) as a pair of simultaneous equations in $U_{i-1}''(y)$ and $V_{i-1}'(y)$ it can be shown that for $i = 1 \dots N+1$

$$\begin{aligned}
 U_{i-1}''(y) = & \frac{h_i^2}{420\tilde{E}_t^i} \left[49P_{i-1}''''(y) + 14P_i''''(y) - 9Q_i''''(y) \right] + \frac{P_{i-1}''(y)}{2\mu_a^i} \\
 & + P_i''(y) \left[\frac{1}{2\mu_a^i} - \frac{v_a^i}{\tilde{E}_A^i} + Q_i''(y) \right] \left[\frac{1}{5\mu_a^i} + \frac{2v_a^i}{5\tilde{E}_A^i} + \frac{2}{h_i^2\tilde{E}_A^i} Q_i(y) - \frac{h_i}{2\tilde{E}_t^i} S_{i-1}''(y) - \varepsilon \right]
 \end{aligned} \quad (82)$$

and

$$\begin{aligned}
 V_{i-1}'(y) = & \frac{h_i^3}{2520\tilde{E}_t^i} \left[63P_{i-1}''''(y) + 21P_i''''(y) - 13Q_i''''(y) \right] \\
 & + \frac{h_i}{60} \left[\frac{1}{\mu_a^i} - \frac{2v_a^i}{\tilde{E}_A^i} \right] \left[5P_i''(y) - 5P_{i-1}''(y) - Q_i''(y) \right] + \frac{1}{h_i\tilde{E}_A^i} \left[P_{i-1}(y) - P_i(y) + Q_i(y) \right] \\
 & - \frac{v_a^i}{\tilde{E}_A^i} S_{i-1}(y) - \frac{h_i^2}{12\tilde{E}_t^i} S_{i-1}''(y) + \varepsilon + \varepsilon x_{i-1} .
 \end{aligned} \quad (83)$$

It follows from (58), (61), (63) and (64) that

$$U_{i-1}^{0''}(y) = -\varepsilon, \quad V_{i-1}^{0'}(y) = \varepsilon + \varepsilon x_{i-1}, \quad i = 1 \dots N+1. \quad (84)$$

The integration of (82) and (83) then leads to

$$\begin{aligned}
 \bar{U}_{i-1}(y) = & \frac{h_i^2}{420\tilde{E}_t^i} \left[49P_{i-1}''(y) + 14P_i''(y) - 9Q_i''(y) \right] + \frac{P_{i-1}(y)}{2\mu_a^i} \\
 & + P_i(y) \left[-\frac{v_a^i}{\tilde{E}_A^i} + Q_i(y) \right] \left[\frac{2v_a^i}{5\tilde{E}_A^i} + \frac{2}{h_i^2\tilde{E}_A^i} \int_0^y \int_0^t Q_i(s) ds dt - \frac{h_i}{2\tilde{E}_t^i} S_{i-1}(y) \right], \quad i = 1 \dots N+1
 \end{aligned} \quad (85)$$

where

$$\begin{aligned}
\tilde{V}_{i-1}(y) = & \frac{h_i^3}{2520 \tilde{E}_t^i} \left[63 P_{i-1}'''(y) + 21 P_i'''(y) - 13 Q_i'''(y) \right] \\
& + \frac{h_i}{60} \left[\frac{1}{\mu_a^i} - \frac{2 \vartheta_a^i}{\tilde{E}_A^i} \right] \left[5 P_i'(y) - 5 P_{i-1}'(y) - Q_i'(y) \right] \\
& + \frac{1}{h_i \tilde{E}_A^i} \left[\int_0^y (P_{i-1}(s) - P_i(s) + Q_i(s)) ds \right] - \frac{\vartheta_a^i}{\tilde{E}_A^i} \int_0^y S_{i-1}(s) ds \\
& - \frac{h_i^2}{12 \tilde{E}_t^i} S_{i-1}'(y), \quad i = 1 \dots N+1,
\end{aligned} \tag{86}$$

$$\tilde{U}_i(y) = U_i(y) - U_i^0(y), \quad \tilde{V}_i(y) = V_i(y) - V_i^0(y), \quad i = 0 \dots N. \tag{87}$$

7.4 Solving the recurrence relations.

From the recurrence relation (43) the following expression can be derived for $S_i(y)$ on using the boundary condition (26)₁ that $\sigma_{xx} = 0$ on $x = x_0 = 0$

$$S_i(y) = \frac{1}{6} \sum_{j=1}^i h_j \left[Q_j''(y) - 3 P_{j-1}''(y) - 3 P_j''(y) \right] \quad i = 1 \dots N, \quad S_0(y) = 0. \tag{88}$$

During the bending of the laminate the external surface displacements $U_0(y)$ and $V_0(y)$ are non-zero. In order to operate the recurrence relations (44) and (45) it is necessary to define displacement differences in terms of the displacements on $x = x_0 = 0$ as follows:

$$\Delta \tilde{U}_i(y) = \tilde{U}_i(y) - \tilde{U}_0(y), \quad i = 0 \dots N, \tag{89}$$

and

$$\Delta \tilde{V}_i(y) = \tilde{V}_i(y) - \tilde{V}_0(y), \quad i = 0 \dots N. \tag{90}$$

The following expression for $\Delta \tilde{U}_i''(y)$ can then be generated from recurrence relation (44)

$$\begin{aligned}
\Delta \tilde{U}_i''(y) = & \sum_{j=1}^i \left[\frac{h_j^2}{12 \tilde{E}_t^j} (Q_j''''(y) - 2 P_j''''(y) - 4 P_{j-1}''''(y)) \right. \\
& \left. + \frac{\vartheta_a^j}{\tilde{E}_A^j} (P_{j-1}''(y) - P_j''(y)) + \frac{h_j}{\tilde{E}_t^j} S_{j-1}''(y) \right] \quad i = 1 \dots N, \quad \Delta \tilde{U}_0''(y) = 0.
\end{aligned} \tag{91}$$

The recurrence relation (45) provides the following expression for $\Delta \tilde{V}_i'(y)$

$$\begin{aligned}
\Delta \tilde{V}'_i(y) = & \sum_{j=1}^i \frac{h_j^3}{120 \tilde{E}_t^j} \left[3Q_j''''(y) - 5P_j''''(y) - 15P_{j-1}''''(y) \right] \\
& + \frac{h_j}{6} \frac{1}{\mu_a^j} \left(3P_{j-1}''(y) + 3P_j''(y) - Q_j''(y) \right) + \frac{v_a^j}{\tilde{E}_A^j} \left(3P_{j-1}''(y) - 3P_j''(y) + Q_j''(y) \right) \quad (92) \\
& \left. \frac{h_j^2}{2\tilde{E}_t^j} S_{j-1}''(y) - h_j \tilde{U}_{j-1}''(y) \right], \quad i = 1 \dots N, \quad \Delta \tilde{V}'_0(y) = 0
\end{aligned}$$

The relations, (91) and (92) will provide $2N$ equations for the determination of the $2N+1$ unknown functions $P_i(y)$, $i = 1 \dots N$, and $Q_i(y)$, $i = 1 \dots N+1$. A further equation is required which is obtained by imposing the boundary condition (26)₂ that σ_{xx} must be zero on $x = x_{N+1}$ for all values of y . From (38) and (88) it follows that

$$\sum_{j=1}^{N+1} h_j \left[Q_j''(y) - 3P_{j-1}''(y) - 3P_j''(y) \right] = 0. \quad (93)$$

Upon rearranging (93) and integrating twice with respect to y

$$Q_{N+1}(y) = \frac{1}{h_{N+1}} \left[3 \sum_{j=1}^{N+1} h_j (P_{j-1}(y) + P_j(y)) - \sum_{j=1}^N h_j Q_j(y) \right]$$

This relation is used to eliminate the function $Q_{N+1}(y)$ in terms of the other $2N$ unknown functions, remembering from (37) that $P_0(y) \equiv P_{N+1}(y) \equiv 0$.

The bending moment per unit width M about the mid-plane of the laminate is given by

$$M = \frac{1}{2} \sum_{i=1}^{N+1} \int_{x_{i-1}}^{x_i} (2x - h) \sigma_{yy}^i dx = \sum_{i=1}^{N+1} \int_{x_{i-1}}^{x_i} x \sigma_{yy}^i dx - \frac{h}{2} \sum_{i=1}^{N+1} \int_{x_{i-1}}^{x_i} \sigma_{yy}^i dx.$$

On integrating (39) it can be shown that

$$\sum_{i=1}^{N+1} \int_{x_{i-1}}^{x_i} \sigma_{yy}^i dx = \sum_{i=1}^{N+1} P_{i-1}(y) - P_i(y) + \int_{x_{i-1}}^{x_i} \sigma_i(x) dx$$

On using (37) and (53), the relation (96) reduces to the expected form

$$\sum_{i=1}^{N+1} \int_{x_{i-1}}^{x_i} \sigma_{yy}^i dx = h \sigma$$

Substituting (39) and (97) into (95) and integrating the appropriate terms it can be shown that

$$M = \sum_{i=1}^{N+1} \left(P_{i-1}(y) - P_i(y) \right) \left(x_i - \frac{h_i}{2} \right) - \frac{h_i}{6} Q_i(y) + \int_{x_{i-1}}^{x_i} x \sigma_i(x) dx - \frac{h^2}{2} \sigma . \quad (98)$$

On using the integrated form of (93) together with (98)

$$M = P_0(y)x_0 - P_{N+1}(y)x_{N+1} + \sum_{i=1}^{N+1} \int_{x_{i-1}}^{x_i} x \sigma_i(x) dx - \frac{h^2}{2} \sigma , \quad (99)$$

which on using (37) reduces to

$$M = \frac{1}{2} \sum_{i=1}^{N+1} \int_{x_{i-1}}^{x_i} (2x - h) \sigma_i(x) dx . \quad (100)$$

This expression is independent of y and the degree of ply cracking, and thus the bending moment does not vary along the length of the laminate, as expected for the case when the upper and lower surfaces of the laminate are stress-free.

7.5 The differential equations

Define functions $\Omega_i(y)$ and $\Theta_i(y)$ to be

$$\Omega_i(y) = \tilde{U}_i''(y) - \tilde{U}_0''(y) - \Delta \tilde{U}_i''(y) = 0 , \quad i = 1 \dots N , \quad (101)$$

where $\tilde{U}_i''(y)$ is derived from (85), and $\Delta \tilde{U}_i''(y)$ is given by (91), and

$$\Theta_i(y) = \tilde{V}_i'(y) - \tilde{V}_0'(y) - \Delta \tilde{V}_i'(y) = 0 , \quad i = 1 \dots N , \quad (102)$$

where $\tilde{V}_i'(y)$ is derived from (86), and $\Delta \tilde{V}_i'(y)$ is given by (92). From the definitions (89) and (90) it is clear that $\Omega_i(y)$ and $\Theta_i(y)$ are both identically zero. Thus on eliminating the function $Q_{N+1}(y)$ using (94), the following system of fourth order differential equations is obtained:

$$\begin{aligned} \sum_{i=1}^N F_{ij} P_i''''(y) + \sum_{i=1}^N G_{ij} P_i''(y) + \sum_{i=1}^N H_{ij} P_i(y) \\ + \sum_{i=1}^N F_{i+Nj} Q_i''''(y) + \sum_{i=1}^N G_{i+Nj} Q_i''(y) + \sum_{i=1}^N H_{i+Nj} Q_i(y) = 0 , \quad j = 1 \dots N , \end{aligned} \quad (103)$$

and

$$\begin{aligned} & \sum_{i=1}^N F_{ij+N} P_i'''(y) + \sum_{i=1}^N G_{ij+N} P_i''(y) + \sum_{i=1}^N H_{ij+N} P_i(y) \\ & + \sum_{i=1}^N F_{i+Nj+N} Q_i'''(y) + \sum_{i=1}^N G_{i+Nj+N} Q_i''(y) + \sum_{i=1}^N H_{i+Nj+N} Q_i(y) = 0, \quad j = 1 \dots N \end{aligned}$$

These differential equations are solved using standard techniques, as described in [5,6] for the case of axisymmetric and laminate stress transfer problems.

8. Application of the boundary conditions

The functions $P_i(y)$ and $Q_i(y)$, for $i=1 \dots N$, that are required to specify the stress and displacement fields of a cracked laminate are now determined as the solution of the $2N$ differential equations (103) and (104). To ensure a unique solution, $8N$ boundary conditions must be imposed.

By imposing $(27)_2$ and (28), $4N$ of the boundary conditions are found. It follows from (34) that $\sigma_{xy}^i = 0$ on $y = \pm L$ for all x only if

$$\begin{aligned} P_i'(\pm L) &= 0, \quad i = 1 \dots N, \\ Q_i'(\pm L) &= 0, \quad i = 1 \dots N. \end{aligned} \quad (106)$$

Suppose there are m CRACKED 90° layers for which $(27)_1$ is to be satisfied for all x . The application of these conditions leads to the following $4m$ boundary conditions, derived from (39) and (51),

$$Q_i(\pm L) = \frac{1}{2} h_i^2 \bar{E}_A^i \bar{\epsilon},$$

$$\frac{1}{h_i \bar{E}_A^i} [P_i(\pm L) - P_{i-1}(\pm L)] = \epsilon - \alpha_A^i \Delta T^i + \left(x_i - \frac{h_i}{2} \right) \bar{\epsilon}$$

The remaining required boundary conditions are found by demanding that the averaged and integrated moment of (29) are satisfied for the $N + 1 - m$ uncracked layers. It can be shown, from (65), that

$$\bar{v}_i(\pm L) = \pm L \left[\left(x_i - \frac{h_i}{2} \right) \bar{\epsilon}_c \right], \quad (109)$$

and from (75) that

$$\hat{v}_i(\pm L) = \pm \frac{h_i L}{2} \left[\epsilon_c + \left(x_i - \frac{h_i}{3} \right) \hat{\epsilon}_c \right] \quad (110)$$

where from (69) and (79), on substituting for $U_i(y)$ and $V_i(y)$ using (85-87), $\bar{v}_i(y)$ and $\hat{v}_i(y)$ can be expressed as

$$\begin{aligned} \bar{v}_i(y) = & \frac{h_i}{12} \frac{\hat{v}_a^i}{E_A^i} \left[4P'_{i-1}(y) + 2P'_i(y) - Q'_i(y) \right] + \frac{1}{h_i \hat{E}_A^i} \left[P_{i-1}^*(y) - P_i^*(y) \right] \\ & - \frac{v_a^i}{E_A^i} S_{i-1}^*(y) + \epsilon y + \hat{\epsilon} \left(x_i - \frac{h_i}{2} \right) y, \quad i = 1 \dots N+1, \end{aligned} \quad (111)$$

$$\begin{aligned} \hat{v}_i(y) = & \frac{h_i^2}{120} \frac{\hat{v}_a^i}{E_A^i} \left[25P'_{i-1}(y) + 15P'_i(y) - 7Q'_i(y) \right] \\ & + \frac{1}{6 \hat{E}_A^i} \left[3P_{i-1}^*(y) - 3P_i^*(y) - Q_i^*(y) \right] \\ & - \frac{h_i}{2} \frac{v_a^i}{E_A^i} S_{i-1}^*(y) + \frac{h_i}{2} \epsilon y + \frac{h_i}{6} \hat{\epsilon} (3x_i - h_i) y, \quad i = 1 \dots N, \end{aligned} \quad (112)$$

where

$$P_i^*(y) = \int_0^y P_i(s) ds, \quad Q_i^*(y) = \int_0^y Q_i(s) ds, \quad S_i^*(y) = \int_0^y S_i(s) ds, \quad i = 1 \dots N+1. \quad (113)$$

Because of (88), (105) and (106), it can be shown from (109-112) that the boundary conditions for uncracked layers are of the form

$$\frac{2Q_i^*(\pm L)}{h_i^2 \hat{E}_A^i L} = \pm (\hat{\epsilon}_c - \hat{\epsilon}), \quad (114)$$

$$\frac{1}{h_i \hat{E}_A^i L} \left[P_{i-1}^*(\pm L) - P_i^*(\pm L) \right] + \frac{2Q_i^*(\pm L)}{h_i^2 \hat{E}_A^i L} \left(x_i - \frac{h_i}{2} \right) = \pm (\epsilon_c - \epsilon). \quad (115)$$

The parameters ϵ_c and $\hat{\epsilon}_c$ are not known in advance and need to be eliminated. This is achieved by subtracting the j^{th} relations (114) and (115) from the others, yielding the following $4(N - m)$ conditions for UNCRACKED layers

$$\begin{aligned}
& \frac{1}{h_i \tilde{E}_A^i} [P_{i-1}^*(\pm L) - P_i^*(\pm L)] - \frac{1}{h_j \tilde{E}_A^j} [P_{j-1}^*(\pm L) - P_j^*(\pm L)] \\
& + \frac{Q_i^*(\pm L)}{h_i^2 \tilde{E}_A^i} (2x_i - h_i) - \frac{Q_j^*(\pm L)}{h_j^2 \tilde{E}_A^j} (2x_j - h_j) = 0,
\end{aligned} \tag{116}$$

$$\frac{Q_i^*(\pm L)}{h_i^2 \tilde{E}_A^i} = \frac{Q_j^*(\pm L)}{h_j^2 \tilde{E}_A^j},$$

where the j^{th} layer is selected to be any one of the uncracked layers.

It is convenient to combine the boundary conditions for the cracked and the uncracked layers into a single set of $2N$ boundary conditions. Upon introducing a new dimensionless parameter λ_i , which takes the value 1 if the i^{th} layer is cracked and 0 otherwise, the conditions (107), (108), (116) and (117) are incorporated as follows

$$\begin{aligned}
& \frac{1 - \lambda_i}{L} \left[\frac{1}{h_i \tilde{E}_A^i} (P_{i-1}^*(\pm L) - P_i^*(\pm L)) - \frac{1}{h_j \tilde{E}_A^j} (P_{j-1}^*(\pm L) - P_j^*(\pm L)) \right] \\
& + \frac{1 - \lambda_i}{L} \left[\frac{Q_i^*(\pm L)}{h_i^2 \tilde{E}_A^i} (2x_i - h_i) - \frac{Q_j^*(\pm L)}{h_j^2 \tilde{E}_A^j} (2x_j - h_j) \right] \\
& + \frac{\lambda_i}{h_i \tilde{E}_A^i} [P_i(\pm L) - P_{i-1}(\pm L)] = \lambda_i \left[\varepsilon - \alpha_A^i \Delta T^i + \left(x_i - \frac{h_i}{2} \right) \varepsilon \right]
\end{aligned}$$

for $i = 1 \dots N+1$, $i \neq j$

$$\frac{1 - \lambda_i}{L} \left[\frac{Q_i^*(\pm L)}{h_i^2 \tilde{E}_A^i} - \frac{Q_j^*(\pm L)}{h_j^2 \tilde{E}_A^j} \right] + \lambda_i \frac{Q_i(\pm L)}{h_i^2 \tilde{E}_A^i} = \frac{\lambda_i}{2} \varepsilon$$

for $i = 1 \dots N+1$, $i \neq j$

The relations (105), (106), (118) and (119) thus provide the necessary $8N$ boundary conditions for a unique solution of the differential equations (103) and (104).

9. The modulus of flexural rigidity

The well known Bernoulli-Euler law for the bending of a homogeneous beam states that

$$\frac{1}{R} = \frac{M}{EI} , \quad (120)$$

where R is the radius of curvature of the neutral axis of the beam, M is the applied bending moment per unit width, E is Young's modulus and I is the second moment of cross sectional area. The product EI provides a measure of the flexural rigidity of the beam, and is called the modulus of flexural rigidity which can be expressed, for the case of the bending of a cracked laminate, as

$$EI = \frac{M}{\varepsilon_c} , \quad (121)$$

where, from (114) assuming symmetry about $y = 0$

$$\varepsilon_c = \varepsilon - \frac{2 Q_j^*(L)}{h_j^2 \tilde{E}_A^j L} \quad (122)$$

where j is the label of the previously selected uncracked layer. The quantity EI was called the flexural modulus in an associated publication [3] but this terminology will not be used here as such terminology is to be defined below in a way that more naturally relates to a modulus.

10. Thermoelastic constants of cracked laminate

When considering temperature changes applied to a cross-ply laminate that is unbalanced and/or contains unsymmetrical ply cracking in the 90° plies, an out-of-plane deformation occurs that is characterised by the flexural thermal expansion coefficient. Because linear elastic deformation is assumed, the relationships defining the thermoelastic behaviour of the laminate are

$$\varepsilon_c = A \sigma + B M + P \Delta T , \quad (123)$$

$$h \varepsilon_c = C \sigma + D M + Q \Delta T , \quad (124)$$

where A , B , C , D , P and Q are the effective thermo-elastic constants for a laminate subject to plane strain conditions. The following identifications are then made

$$A = \frac{1}{\tilde{E}_A^c} , \quad D = \frac{12}{h^2 E_A^c} \quad P = \alpha_A^c , \quad (125)$$

where $\tilde{E}_A^c$ and α_A^c are the axial reduced Young's modulus and thermal expansion coefficient for the cracked laminate, and where E_A^c is the corresponding flexural modulus. The parameters B , C and Q are written

$$B = -\frac{6}{h^2} \quad C = S_n^c, \quad Q = \alpha_n^c$$

The relations (123) and (124) may then be expressed as

$$\varepsilon_c = \frac{\sigma}{\tilde{E}_A^c} - \frac{6M}{h^2 \tilde{E}_n^c} + \alpha_A^c \Delta T$$

$$h\hat{\varepsilon}_c = S_n^c \sigma + \frac{12M}{h^2 \tilde{E}_n^c} + \alpha_n^c \Delta T$$

When the laminate is homogeneous and uncracked, and possibly divided into layers, so that

$$\tilde{E}_A^i = \tilde{E}_A, \quad \alpha_A^i = \alpha_A, \quad \text{and } \Delta T_i = \Delta T, \quad \text{for all } i,$$

it can be shown that

$$\tilde{E}_A^c = \tilde{E}_A, \quad \tilde{E}_n^c = \tilde{E}_A, \quad S_n^c = 0, \quad \alpha_n^c = 0,$$

indicating why the identifications (125) and (126) have been made. By comparing (121) and (128) it is clear that $\tilde{E}_n^c$ is related to the modulus of rigidity EI for a cracked laminate. The relations (127) and (128) are equivalent to those defined for laminate analysis where stress resultants and curvature are used in place of the effective axial stress σ and $\hat{\varepsilon}$.

It is useful to introduce precise terminology for the thermo-elastic constants appearing in the relations (127) and (128). The various laminate constants are labelled as follows:

- $\tilde{E}_A^c$ - axial Young's modulus of cracked laminate for plane strain conditions,
- $\tilde{E}_n^c$ - flexural modulus for uniaxial loading of cracked laminate,
- S_n^c - uniaxial compliance for bending of cracked laminate,
- $\tilde{E}_n^c$ - flexural modulus for bending of cracked laminate,
- α_A^c - axial thermal expansion coefficient of cracked laminate for plane strain conditions,
- α_n^c - flexural thermal expansion coefficient of cracked laminate.

11. RESULTS OF PREDICTIONS

The new model of stress transfer that can account for the effects of bending and thermal residual stresses has already been tested [3] by considering a problem for which an accurate solution is already known. A more detailed discussion of these results is now given. The methodology was applied to the tensile deformation (without bending) of a simple symmetric $[0/90_2/0]$ laminate which has been solved using the generalised plane strain stress transfer model described in references [7,8] and using ply refinement techniques to obtain accurate solutions. The materials properties selected for these calculations were for a laminate whose ply properties correspond to a filament wound composite where Silenka E-glass 1200tex fibre

reinforces MY750/HY917/DY063 epoxy [9]

$E_A = 45.6 \text{ GPa}$	$E_T = 16.2 \text{ GPa}$	$\nu_A = 0.278$
$\nu_T = 0.4$	$\mu_A = 5.83 \text{ GPa}$	
$\alpha_A = 8.6 \times 10^{-6}/^\circ\text{C}$	$\alpha_T = 26.4 \times 10^{-6}/^\circ\text{C}$	

As an example not intended to be physically realistic, an axial tensile stress of 1 GPa was applied to the laminate and the temperature change was selected so that $\Delta T = -100^\circ\text{C}$. The thickness of the outer 0° plies was selected to be 0.25 mm while the single inner 90° ply had a total thickness of 0.5 mm. A system of equally spaced ply cracks in the 90° ply was selected so that the crack separation was 5 mm, corresponding to a crack density of 0.2 cracks/mm. When carrying out the generalised plane strain analysis for the special case of plane strain conditions using the methods of [7,8], the 0° plies were each divided into 20 layers having the same properties, while the inner 90° ply was divided into 40 plies. The layers nearest to the interfaces between the 0° and 90° plies were then subdivided successively three times in order to achieve layer refinement near the crack tip singularities. When applying the new model described here, the 0° plies were each divided into 5 layers while the inner 90° ply was divided into 10 layers. Again the layers at the ply interfaces were successively subdivided three times. The results of predictions using the two methods for this special case are shown in Figs.1-3.

Fig.1 shows the axial stress in the plane of a ply crack between the centre of the ply at $x = h/2$ and the outer surface at $x = h$. In the region $0 \leq x - h/2 \leq 0.25$ where the 90° ply is cracked, the axial stress is exactly zero as required. Equivalent finite element calculations do not usually satisfy this boundary condition without some error. For the region $0.25 \leq x - h/2 \leq 0.5$ of the 0° ply, the axial stress is non-uniform exhibiting a singularity at the crack tip $x - h/2 = 0.25$. The step function (thick line) is the prediction based on the generalised plane strain model [7,8] while the thin line corresponds to predictions based upon the new model described here. The axial stress must be discontinuous at ply interfaces. It is seen, however, to be discontinuous also at all layer interfaces, especially as the singularity is approached. In spite of this the two models exhibit very good agreement leading to confidence in the methodology. In the region of the singularity the discontinuity in axial stress can be large (as shown) when using ply refinement. The axial stress could have been averaged across each ply element and results plotted as a step function of averaged stresses.

Fig.2 compares predictions for the axial displacement in the plane of the crack where the crack opening is clearly seen. The crack opening profile predicted by the two methods is almost coincident, but in the uncracked region $x - h/2 \geq 0.25$ where the axial displacement should be uniform, very slight perturbations of the axial displacement are observed for both methods. The exact solution of the elasticity problem would lead to a uniform axial displacement distribution for the uncracked ligament. The fact that the predictions of both models are almost uniform is an indication of the high accuracies that can be achieved by the methodology.

It should be noted that far fewer layers were used to refine the plies when using the new model. This is because the representation used for the stress and displacement fields already has some degree of through-thickness variation. If many more ply elements are used to represent each ply then numerical instabilities are encountered arising from the influence of the stress singularity at the ply crack tip. If such problems arise then they are easily identified and corrected by using a lower level of ply refinement.

Fig.3 compares the predictions of the distributions of the normal and shear interfacial stresses along the length of the interface between the 0° and 90° plies. It is not possible to distinguish between the predictions of the new model and those obtained using the generalised plane strain model.

The new stress transfer model has already been applied [3], using the properties given above, to a symmetric $[0/90/0/90/0]$ laminate where each ply has the same thickness 0.2 mm. A bending moment per unit width of 1 kN was applied together with a uniform temperature difference $\Delta T = -100^\circ\text{C}$. Each ply was subdivided into 5 layers of equal thickness and each layer next to the interfaces between plies was successively subdivided twice. An array of equally spaced ply cracks having separation 1 mm was considered in only one of the 90° plies. Fig.4 shows the resulting predictions of the axial stress on a plane containing one of the ply cracks. It is seen that the axial stress is exactly zero in the region of the crack ($0.6 < x < 0.8$), and that in the uncracked half of the laminate $0 \leq x \leq 0.5$ there appears to be little effect of the cracked 90° ply. This, however, is not the case as the presence of the crack leads to additional bending. This additional bending is such that the axial displacement remains linear in x for the uncracked plies as seen in Fig.5 where the axial displacement distribution in the plane of the crack is plotted. The crack opening profile is clearly seen together with very slight errors near the crack tips where the axial displacement oscillates with a small amplitude rather than being a linear function that would arise from an exact solution.

Having discussed predictions for the stress and displacement distributions in a cracked laminate, it is now appropriate to consider the effect of ply cracking on the thermo-elastic constants appearing in the relations (127) and (128) that characterise the behaviour of any cross-ply laminate subject to combined axial loading and bending, and to uniform temperature variations. The thermo-elastic constants have been calculated as a function of crack density for a GRP cross-ply laminate of the type $[0/90_2/0/90_2/0]$ using the same ply properties as used above for other predictions. The cracks were assumed to be uniformly distributed in just one of the 90° plies. The number of ply elements used to represent each ply in the laminate is denoted by n taking values in the range 1-5, and in each ply the elements were assumed to have the same thickness. There was no ply refinement near the crack tips.

Figs.6-11 shows the predicted values of the effective thermo-elastic constants of the laminate as a function of crack density for values $n = 1 \dots 5$. It is seen in all cases that as ply refinement increases from $n = 1$ to $n = 5$ the predictions are converging to an accurate estimate of the properties for all values of the crack density in the range up to 4 cracks/mm. It is clear from the results that the selection of $n = 1$ (i.e. no ply refinement) leads to results that are significantly different to the accurate solution specified when $n = 5$.

12. CONCLUSIONS

1. The new model of stress transfer, when applied to the axial loading (under plane strain conditions) of a symmetric laminate leads to stress and displacement distributions that are very close to the predictions obtained using a generalised plane strain model, thus demonstrating the accuracy and consistency of both types of model.
2. The application of the model to the bending of a $[0/90/0/90/0]$ laminate where cracks are present in just one of the 90° plies leads to accurate solutions for stress and displacement.
3. The methodology has been applied to the prediction of the dependence of the effective thermo-elastic constants of a cross-ply laminate (having unsymmetrical cracking) on crack density, where it has been shown that ply refinement is needed to improve the accuracy of predictions.

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REFERENCES

1. McCartney, L.N., 'Stress transfer mechanics for ply cracks in general symmetric laminates', NPL Report CMMT(A)50, December 1996.
2. McCartney, L.N., 'Generalised framework for the prediction of ply cracking in any general symmetric laminate subject to general in-plane loading', NPL Report CMMT(A)51, December 1996.
3. McCartney, L.N. & Pierse, C., 'Stress transfer mechanics for multiple ply laminates for axial loading and bending', to appear in Proc. ICCM-10, Gold Coast, Australia, July 1997.
4. Li, S., Reid, S.R., & Soden, P.D., 'A finite strip analysis of cracked laminates', *Mech. of Mater.*, **18**, (1994), 289-311.
5. Hannaby, S.A., 'The solution of ordinary differential equations arising from stress transfer mechanics' NPL Report DITC 223/93, November 1993.
6. Hannaby, S.A., 'The solution of ordinary differential equations arising from stress transfer mechanics of general symmetric laminates' NPL Report CISE 13/97, April 1997.
7. McCartney, L.N., 'A recursive method of calculating stress transfer in multiple-ply cross-ply laminates subject to biaxial loading', NPL Report DMM(A)150, January 1995.
8. McCartney, L.N., 'Predicting transverse crack formation in cross-ply laminates resulting from micro-cracking', to appear in *Comp. Sci & Tech.*.
9. Soden M. and Hinton, M., Data being used in an international intercomparison exercise, the results of which will be published in *Comp. Sci. & Tech.*.

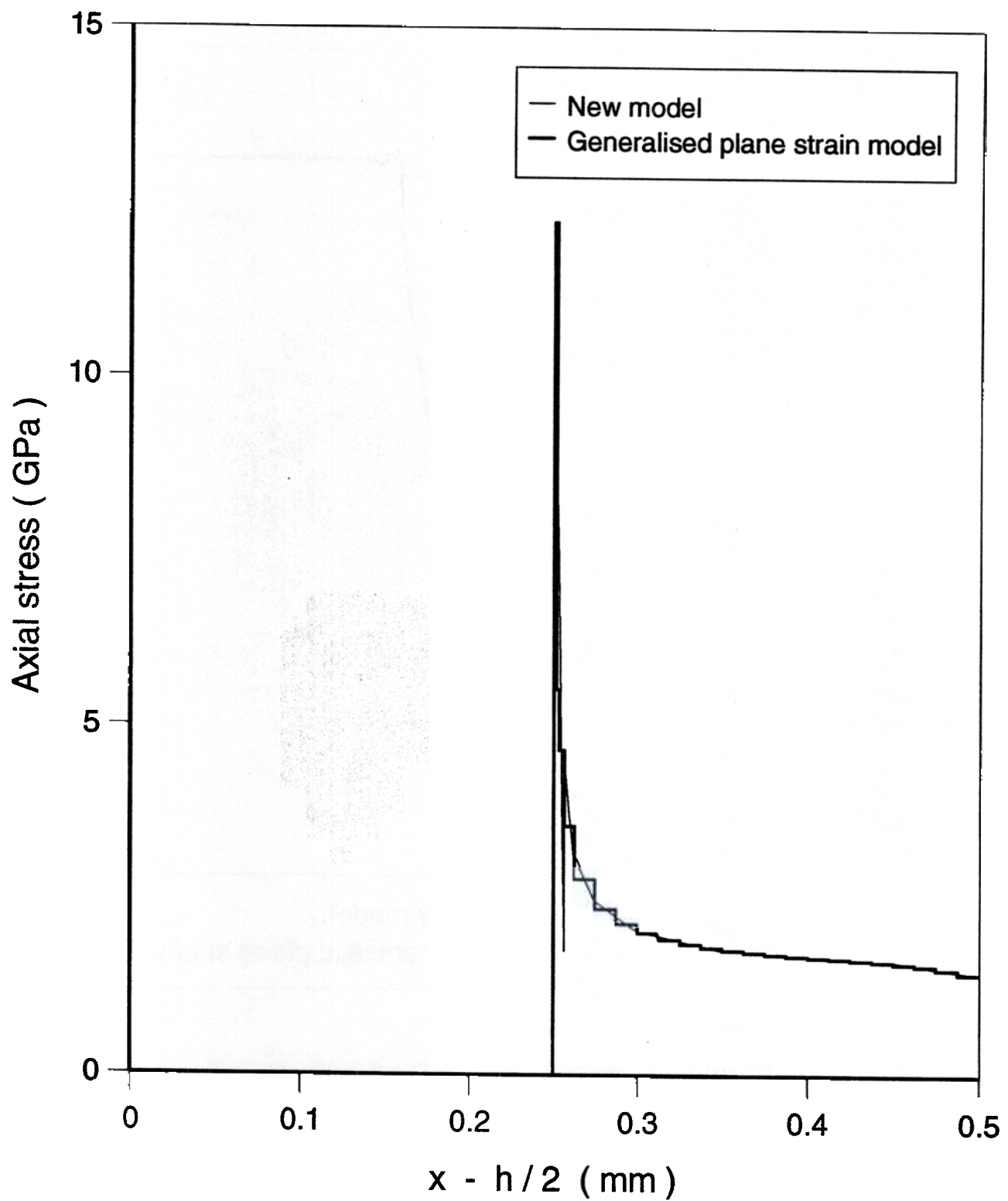


FIG.1 : Comparison of the axial stress distributions in the plane of a ply crack predicted by the new model and that of an existing generalised plane strain model for a cross-ply GRP laminate.

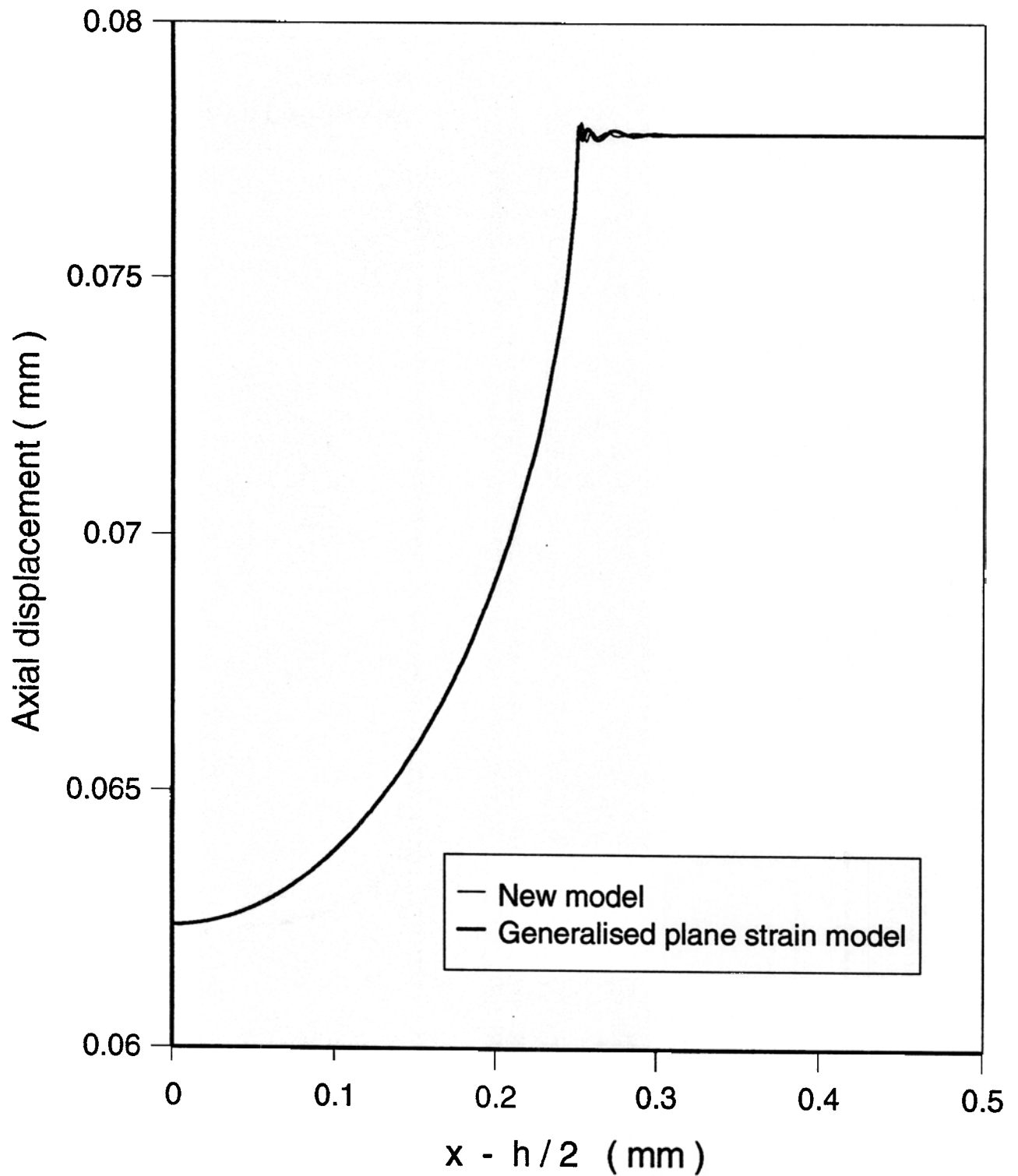


FIG.2 : Comparison of the axial displacement distributions in the plane of a ply crack predicted by the new model and that of an existing generalised plane strain model for a symmetric cross-ply laminate.

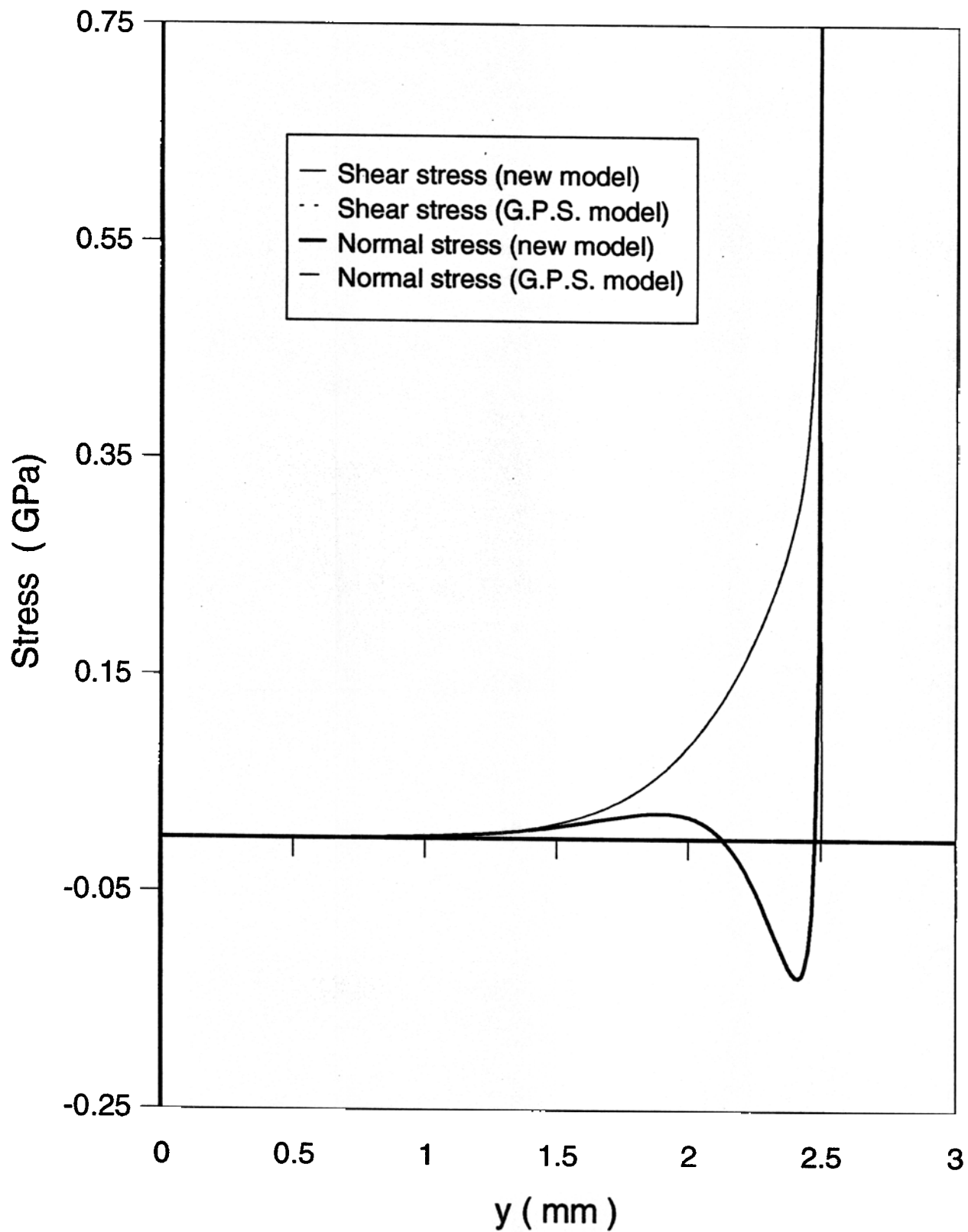


FIG.3 : Comparison of the interfacial shear and normal stresses predicted by the new model and that of an existing generalised plane strain (G.P.S.) model for a cross-ply GRP laminate.

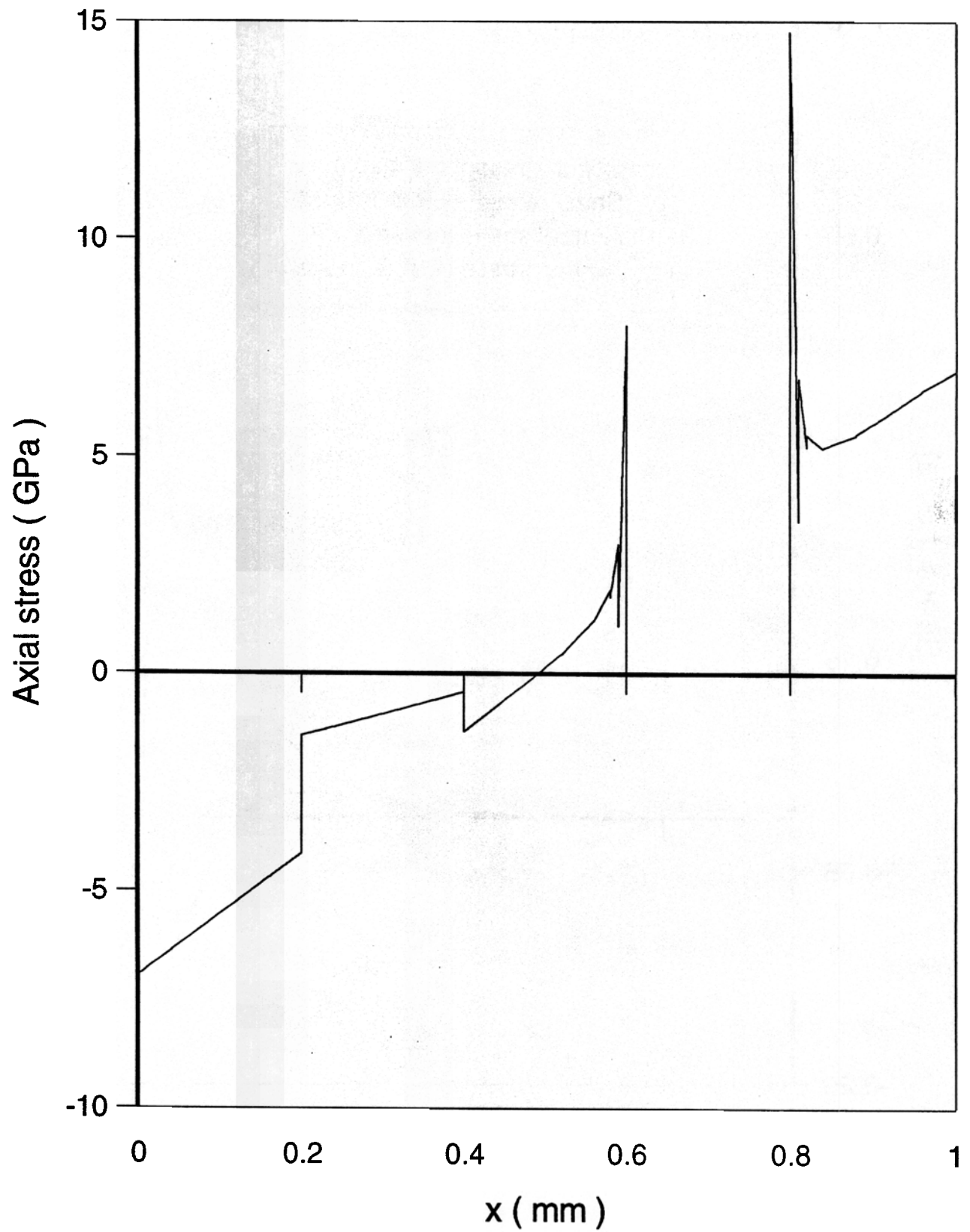


FIG.4 : Axial stress distribution in ply crack plane for the lower 90-ply of a [0/90/0/90/0] GRP laminate.

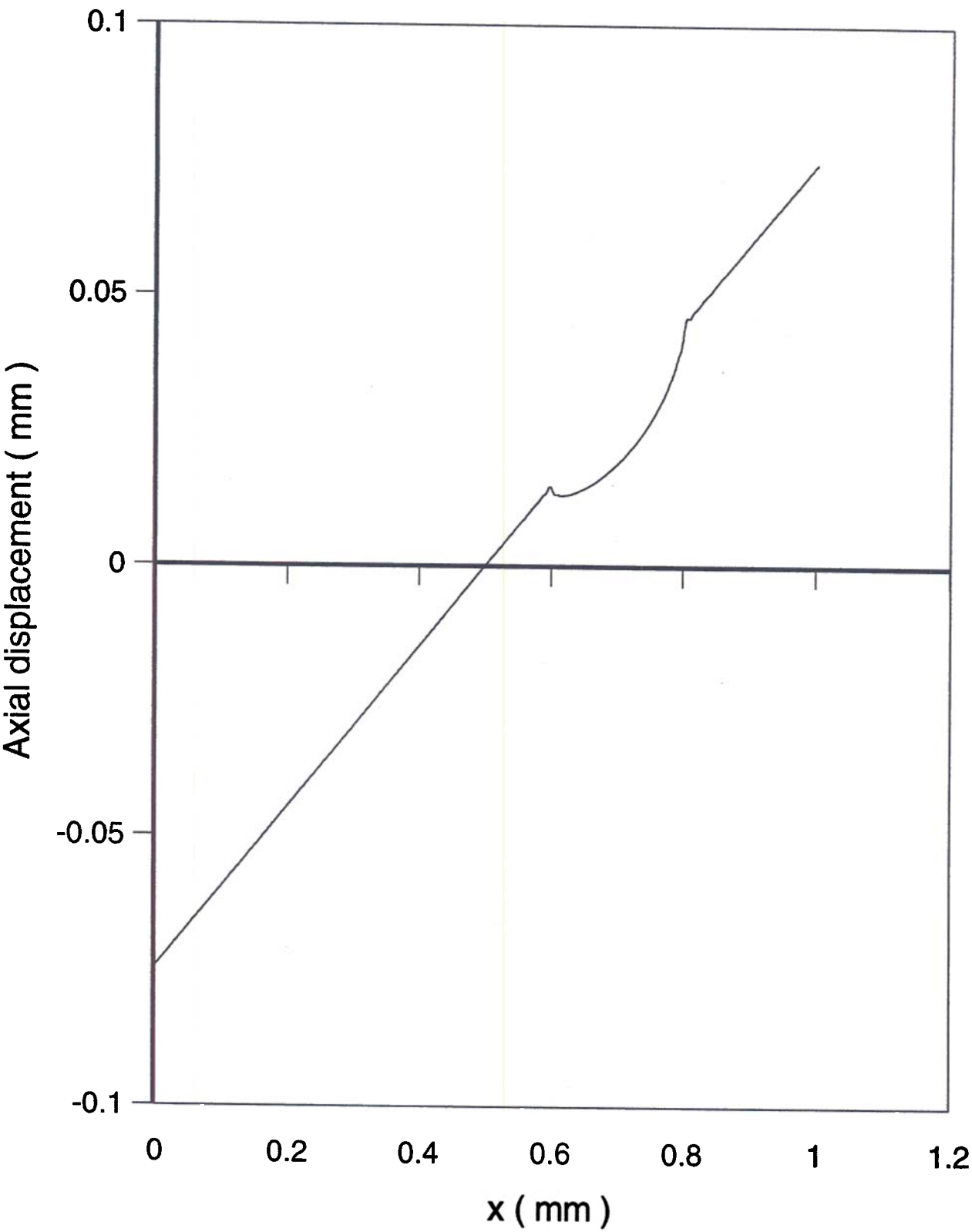


FIG.5 : Axial displacement distribution in the ply crack plane for the lower 90-ply of a [0/90/0/90/0] GRP laminate.

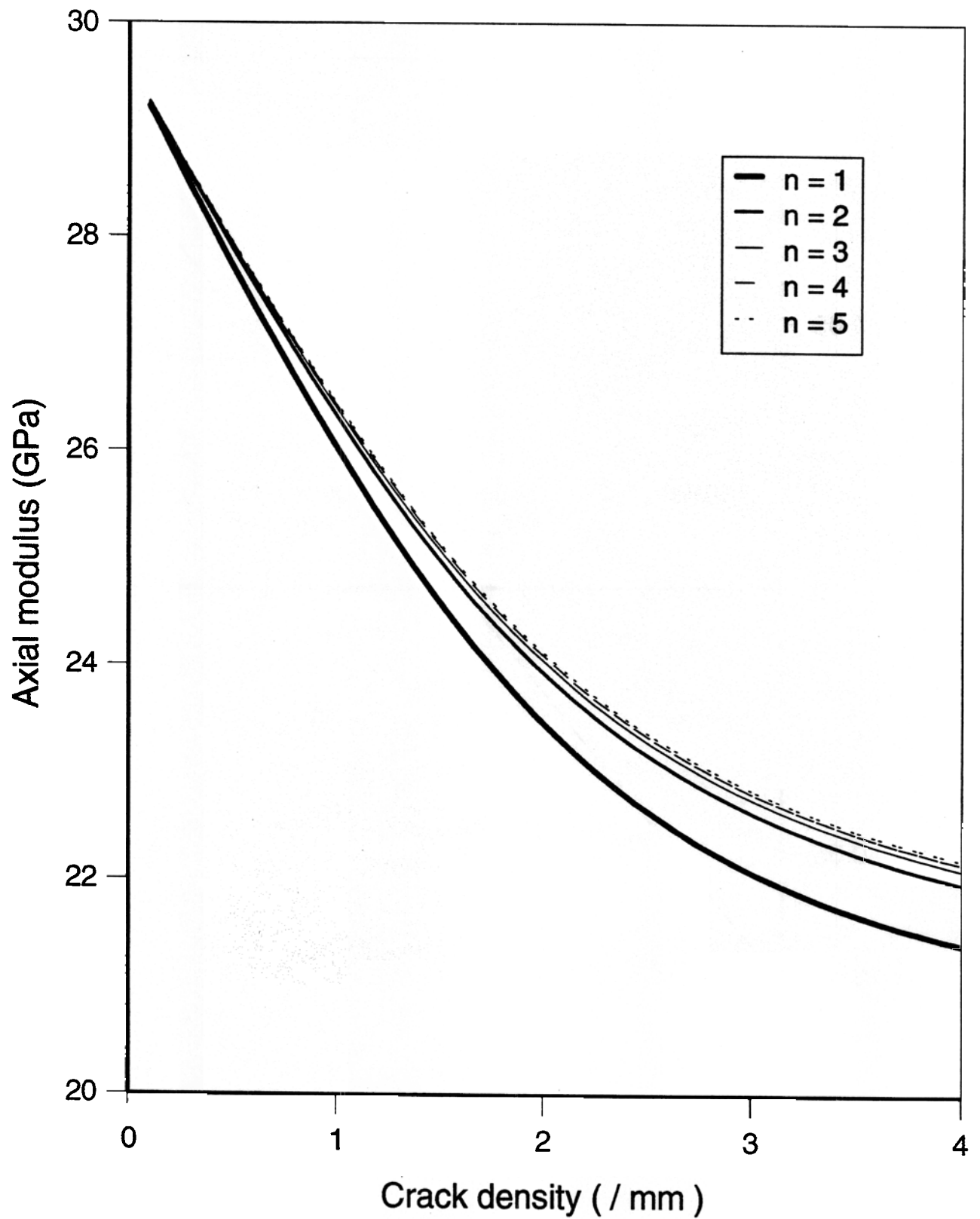


FIG.6 : Dependence of the axial modulus on crack density for a symmetrical GRP cross-ply laminate having non-symmetrical ply cracking. The parameter n denotes the number of layers used to represent a singly ply.

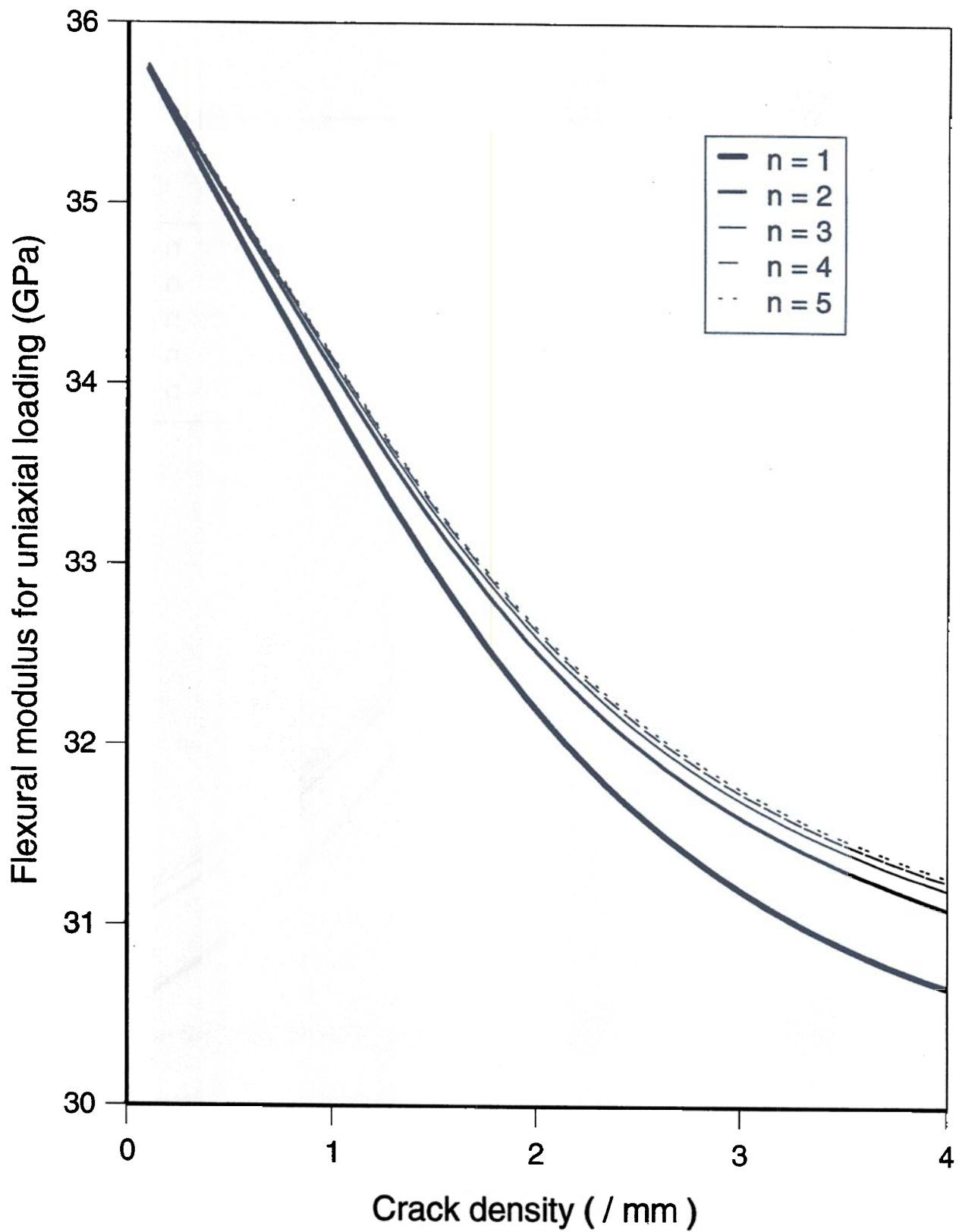


FIG.7 : Dependence of the flexural modulus for uniaxial loading on crack density for a symmetrical GRP cross-ply laminate having non-symmetrical ply cracking. The parameter n denotes the number of layers used to represent a single ply.

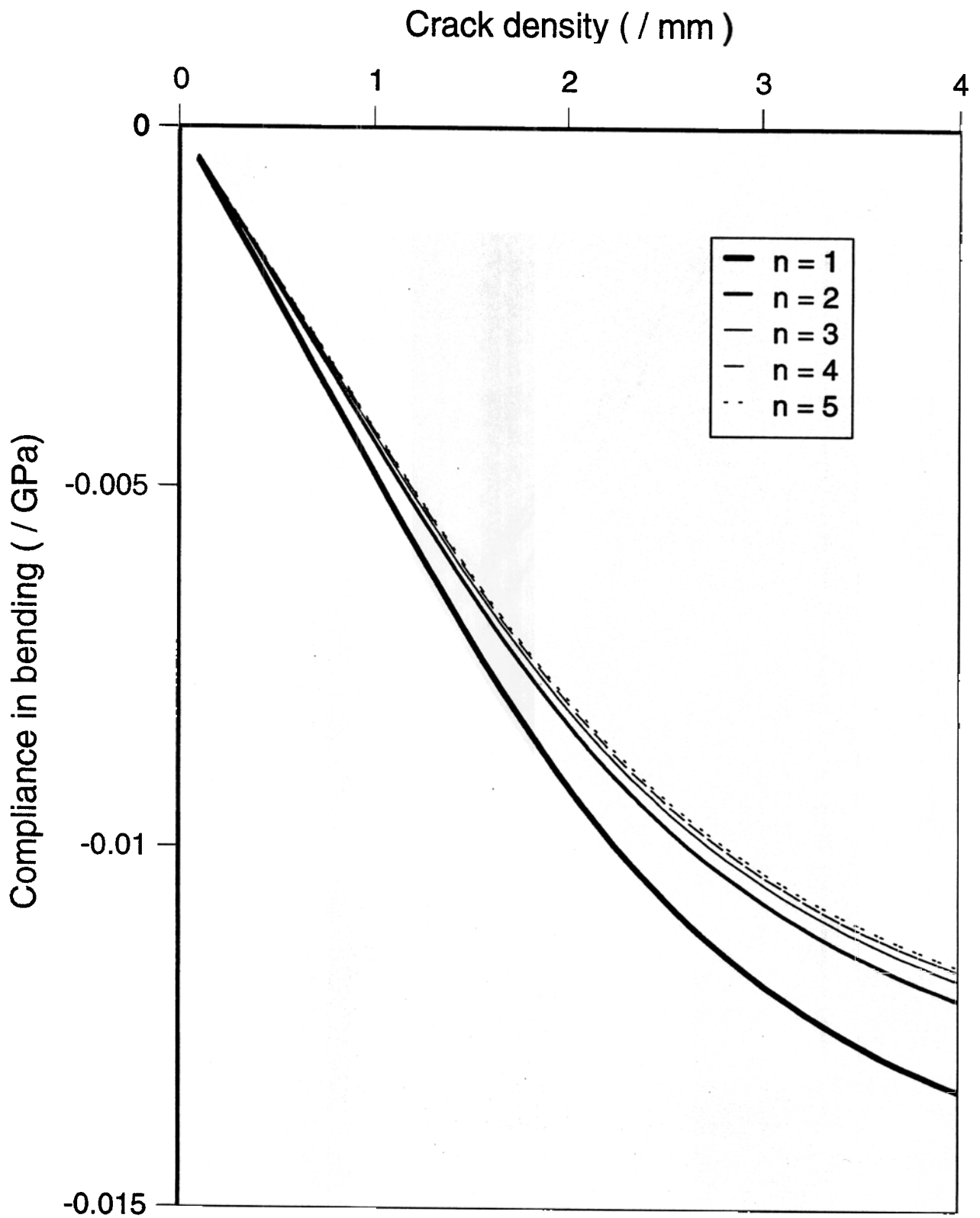


FIG.8 : Dependence of compliance for bending on crack density for a symmetrical GRP cross-ply laminate having non-symmetrical ply cracking. The parameter n denotes the number of layers used to represent a single ply.

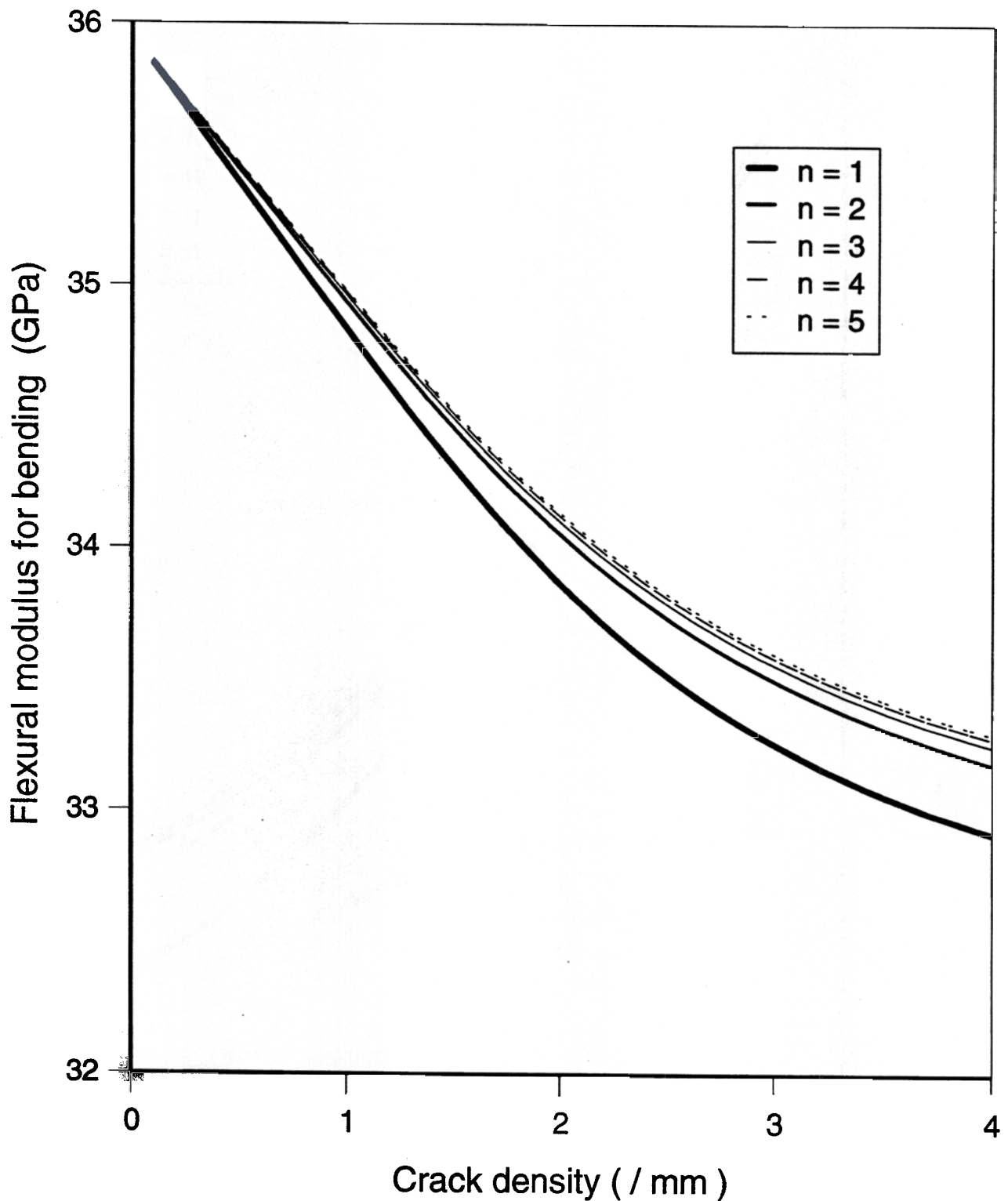


FIG.9 : Dependence of the flexural modulus in bending on crack density for a symmetrical GRP cross-ply laminate having non-symmetrical ply cracking. The parameter n denotes the number of layers used to represent a single ply.

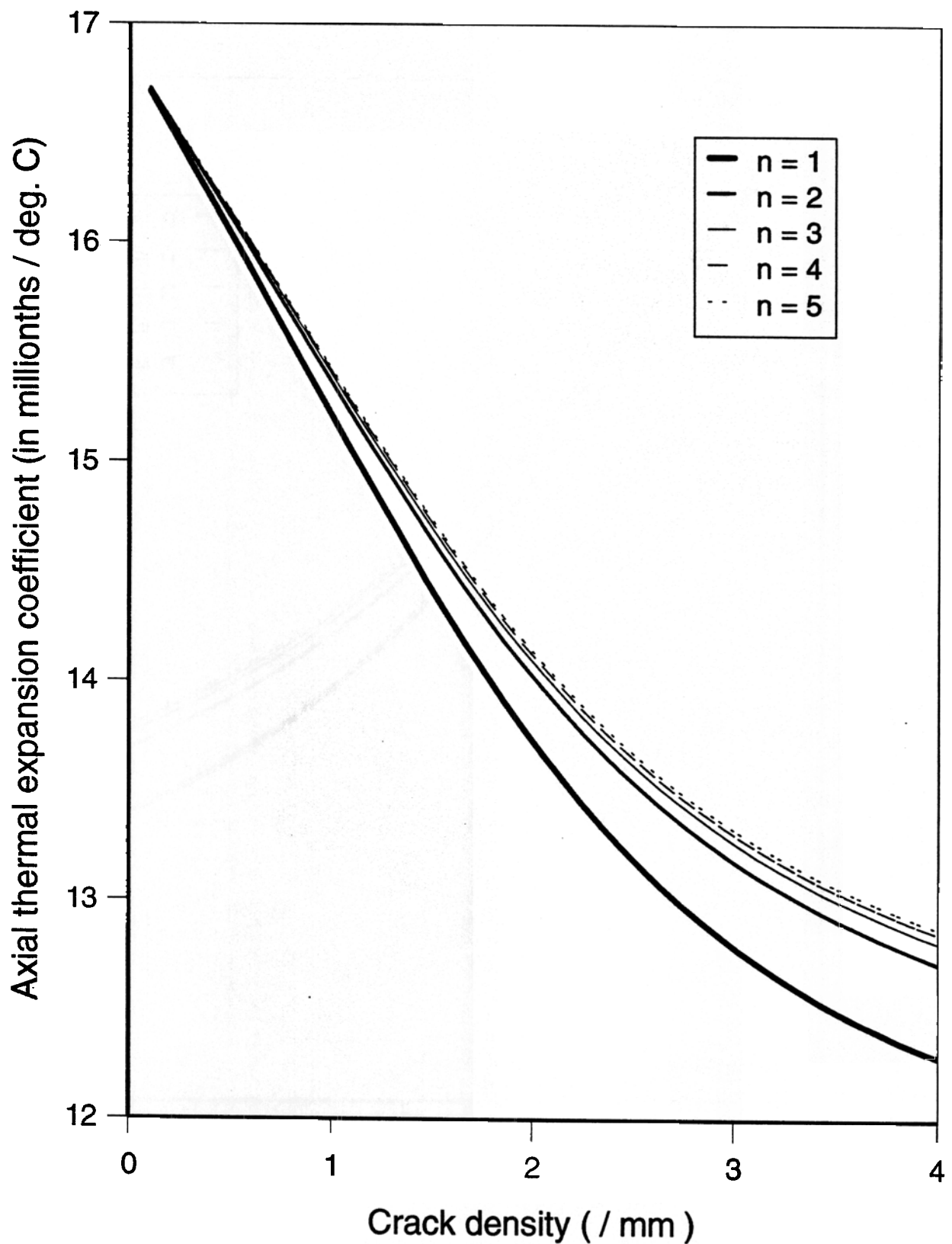


FIG.10 : Dependence of axial thermal expansion coefficient on crack density for a symmetrical GRP cross-ply laminate having non-symmetrical ply cracking. The parameter n denotes the number of layers used to represent a single ply.

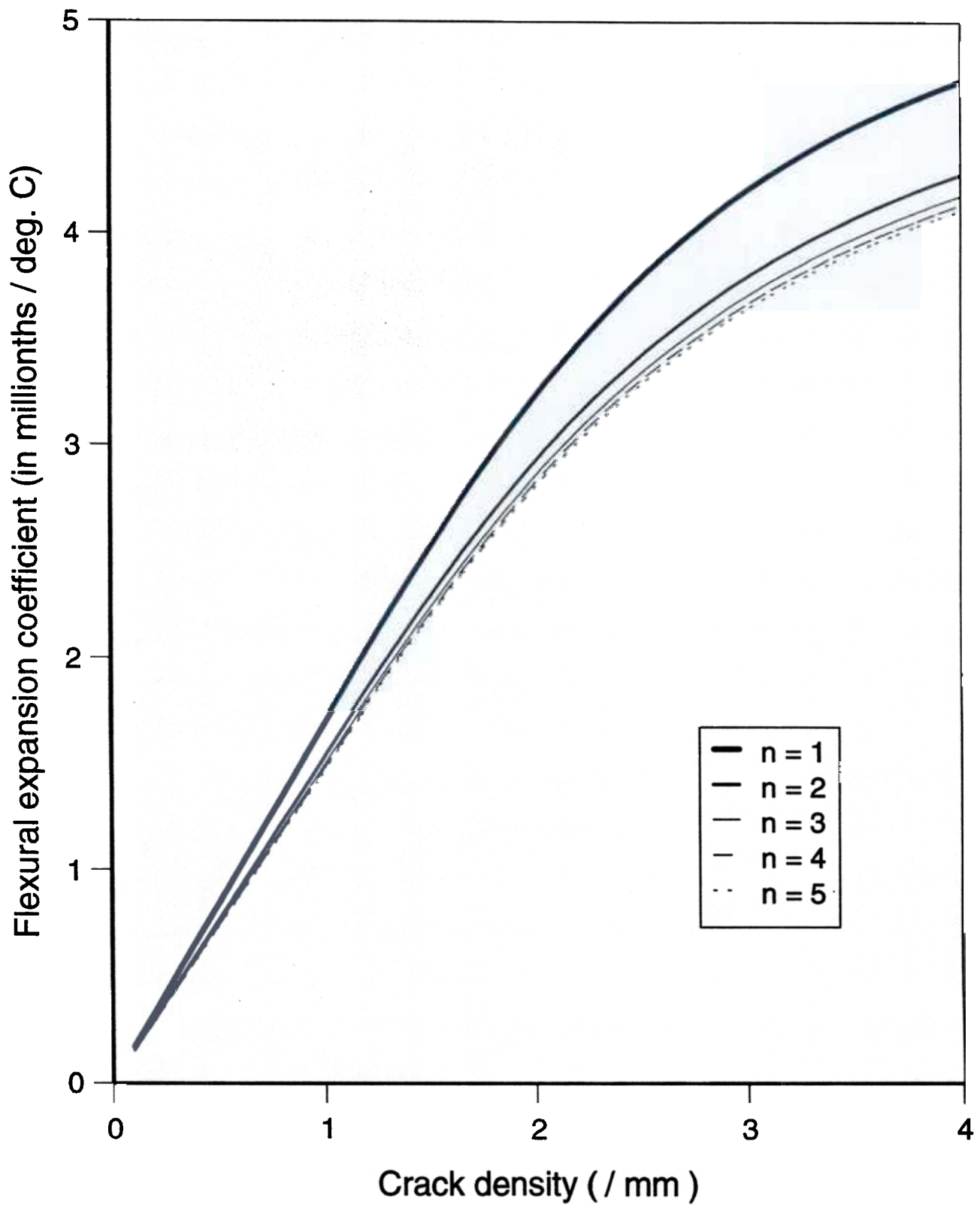


FIG.11 : Dependence of flexural thermal expansion coefficient on crack density for a symmetrical GRP cross-ply laminate having non-symmetrical ply cracking. The parameter n denotes the number of layers used to represent a single ply.