

REPORT

Project CAM6 Development of Test Methods for Design under High Rates of Loading

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Data Requirements for
the Use of Finite Element
Methods to Predict the
Impact Performance of
Plastics**

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Data Requirements for the Use of Finite Element Methods to predict the Impact Performance of Plastics

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SUMMARY

The capability to carry out quantitative predictions of impact performance of plastics parts is becoming increasingly available through the use of finite element analysis. The reliability of these calculations depends on the validity of the materials model used by the analysis for describing the deformation behaviour of polymeric materials. In this report a range of potentially suitable models is reviewed. The data requirements for preferred models are identified, and test methods and data analysis are discussed for the determination of the materials parameters required by these models.

Deformation under impact will generally take the material to large strains where relationships between stress and strain are non-linear. Many of the models used by FE packages to describe non-linear behaviour are elastic-plastic models. These models have been developed for metallic materials and interpret non-linearity as caused by plastic deformation. The shape of the non-linear part of a stress/strain curve is then determined by the strain hardening behaviour of the material. A realistic yield criterion needs to be identified, and it is known that, for plastics, this should be sensitive to the hydrostatic component of stress. In addition, the strain rate under an impact loading will vary with time and position in the component. Since plastics are viscoelastic, the rate dependence of properties will also need to be included in the model.

The determination of parameters that define the yield behaviour of a material that is sensitive to hydrostatic stress requires tests under at least two different states of stress. In this report, it is shown how data from tensile and shear tests can be analysed to give these parameters. In the tensile tests, measurements of lateral and longitudinal strain are needed to generate plots of true stress against plastic strain which characterise the hardening behaviour. These measurements also allow a value for the flow parameter to be derived.

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1 INTRODUCTION

A finite element analysis (FEA) enables the stress and strain distributions to be predicted in a component under load. From these calculations, performance criteria, such as the overall deformation of the component or the energy associated with a level of stress, strain or deformation, can be derived. The value of FEA for design relies on having an appropriate failure criterion, such as a maximum permissible stress, strain or displacement, for comparison with the predictions of the analysis.

A stress analysis by finite element methods requires a materials model that relates stress and strain using suitable mathematical relationships. The selection of this model will depend on the behaviour of the material being analysed. For an analysis of impact performance, large deformations, and hence strains, are typically encountered. The model will therefore need to be able to describe the non-linear behaviour of the material. In addition, strain rates under impact are both high and variable with time and position. Since plastics are viscoelastic, the rate-dependence of properties will therefore also need to be taken into account in the analysis.

In this report, a number of materials models used by FE methods for predicting performance under impact loading are introduced. The relevance of these models for describing the behaviour of plastics materials is considered, and the requirements for property data are identified. In section 5, test methods and procedures for the analysis of measurements are discussed for the determination of the required data.

2 MATERIALS MODELS FOR NON-LINEAR, RATE-DEPENDENT BEHAVIOUR

Many of the models used by FE packages to describe non-linear behaviour are elastic-plastic models. These models have been developed for metallic materials. Linear elastic behaviour is confined to small strains. The onset of non-linearity is interpreted as plastic deformation, and the stress that marks the onset of curvature on a stress/strain curve is considered as the first yield point. The subsequent increase of stress with strain is interpreted as strain hardening. In this region, the total strain comprises both elastic and plastic components, the latter, according to the model, being non recoverable. In order to carry out a stress analysis under stress states that are multiaxial (ie where stress components exist along more than one direction), a knowledge is needed of parameters that define the yield criterion and flow

behaviour. For viscoelastic materials, the parameters that characterise yield, hardening and flow will, in general, be rate dependent.

With polymeric materials, non-linear behaviour is generally associated with enhanced creep caused by an increase in molecular mobility under elevated stresses. This non-linear deformation is recoverable at moderate strains, although the recovery period may be significantly greater than the loading time. Irrecoverable plastic deformation occurs as the peak in the stress/strain curve is approached, and the first maximum in the curve is usually referred to as the yield stress. With polymer blends, some contribution to non-linearity may result below this stress maximum from local shear yielding or crazing around rubber particles.

In the absence of FE models that have been developed specifically for plastics, it is necessary to evaluate which of the available models are best suited to plastics and, where possible, identify limitations in their application.

3 DATA REQUIREMENTS FOR ELASTIC-PLASTIC MATERIALS MODELS

3.1 YIELD CRITERIA

An appropriate yield criterion is a common requirement for all elastic-plastic materials models. A number of criteria have been proposed and evaluated for polymeric materials (1-5). It is apparent from these studies that the yield behaviour of plastics is, in general, sensitive to both the deviatoric (shear) and the hydrostatic components of stress. Yield criteria are therefore usually expressed in terms of stress invariants J_1 and J_{2D} . These invariants are related to principal stress components σ_1 , σ_2 and σ_3 by the equations

$$J_1 = \sigma_1 + \sigma_2 + \sigma_3 \quad (1)$$

$$J_{2D} = \frac{1}{6} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \quad (2)$$

3.1.1 The linear Drucker-Prager yield criterion

A simple yield criterion that has been proposed for plastics (2,5) considers yield stresses to be linearly related to the hydrostatic component of stress J_1 as follows

$$\sigma_T = \frac{\sqrt{3}(\lambda+1)}{2\lambda} J_{2D}^{\frac{1}{2}} + \frac{(\lambda-1)}{2\lambda} J_1 \quad (3)$$

Here σ_T is a yield stress in uniaxial tension and, for a strain hardening material, will increase with the plastic component of tensile strain. For a particular value of σ_T , equation (3) defines a yield surface. This surface will expand as σ_T increases with strain. The parameter λ is a measure of the sensitivity of yield behaviour to the hydrostatic component of stress and is the ratio of equivalent yield stresses in uniaxial compression and uniaxial tension. Equivalent yield stresses are stress levels that lie on the same yield surface (ie that have the same effective plastic strain). If $\lambda=1$, equation (3) reduces to the von Mises yield criterion.

In a principal stress coordinate system, the yield surface given by equation (3) for a particular value of σ_T is a cone with an axis that coincides with the hydrostatic stress axis. As σ_T increases with plastic strain, the cone expands uniformly, assuming λ is constant. If the material obeys von Mises yielding, $\lambda=1$ and the cone degenerates to a cylinder.

The linear Drucker-Prager yield criterion is a hydrostatic stress sensitive yield criterion used in various FE packages. It was developed to describe slip in soil and related materials but is essentially identical to equation (3). The extended version of this criterion takes the form (see Abaqus technical manual)

$$t - p \tan \beta = d \quad (4)$$

$$\text{where } t = \frac{1}{2}q \left[1 + \frac{1}{K} - \left(1 - \frac{1}{K} \right) \left(\frac{r}{q} \right)^3 \right]$$

Here p , q and r are stress invariants and K is a parameter whose magnitude influences the symmetry of the yield surface in the deviatoric plane. No attempt to measure K for plastics is known and $K=1$ seems a valid assumption. Equation (4) then becomes

$$d = q - p \tan \beta \quad (5)$$

This is identical to equation (3) with the following relationships between parameters. It is evident that the von Mises criterion is obtained for $\beta=0$.

$$\begin{aligned}
 q &= (3J_{2D})^{1/2}, & p &= -\frac{J_1}{3} \\
 d &= \frac{2\lambda\sigma_T}{\lambda+1}, & \tan\beta &= \frac{3(\lambda-1)}{\lambda+1}
 \end{aligned}
 \tag{6}$$

3.1.2 The exponent Drucker-Prager yield criterion

This hydrostatic-stress-sensitive yield criterion takes the form (see Abaqus technical manual)

$$aq^b - p - p_t = 0 \tag{7}$$

The criterion contains 3 material parameters, and their determination would involve more complicated testing than required by the previous model (section 3.1.1). For this reason, this model is of little interest at present but it is interesting to note that, for certain values of the parameters, the criterion reduces to an expression that has been used to model yield in plastics (4,6). If

$$\begin{aligned}
 a &= \frac{1}{3(\lambda-1)\sigma_T} \\
 b &= 2
 \end{aligned}
 \tag{8}$$

and

$$p_t = \frac{\lambda\sigma_T}{3(\lambda-1)}$$

equation (7) becomes

$$\lambda\sigma_T^2 = 3J_{2D} + (\lambda-1)\sigma_T J_1 \tag{9}$$

For a strain hardening material σ_T must increase with plastic strain. In axes of principal stress, the yield surfaces defined by equation (9) are a series of paraboloids with a common symmetry axis along the hydrostatic stress direction.

Experimental studies on selected plastics (4,6) have shown that the criteria given by equations (3) and (9) are equally successful in modelling yield under biaxial stresses. Under a triaxial stress, differences between these criteria may be more significant.

3.1.3 The Mohr-Coulomb criterion

The Mohr-Coulomb criterion is a modification of the Tresca yield criterion to include the influence of the hydrostatic stress. It is therefore expressed in terms of maximum and minimum principal stresses, σ_1 and σ_3 respectively, as follows

$$\frac{4\lambda\sigma_T}{(3\lambda+1)} = \sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3) \frac{(\lambda-1)}{(\lambda+1)} \quad (10)$$

For a particular value of σ_T , the yield surface is a hexagonal cone in principal stress axes. Whilst the parameters that need to be determined for this model are the same (not numerically) as the previous models, we know of no reason why this model should be chosen in preference.

3.2 THE FLOW LAW

Each of the yield criteria discussed so far enable an effective (or equivalent) stress to be defined that is a measure of the amount of plastic deformation generated by an applied stress field. All stress states that have the same effective stress lie on the same yield surface. An effective plastic strain $\bar{\epsilon}_p$ can be defined, analogous to the effective stress, in terms of strain invariants (see Appendix 1). The effective stress and strain are related through a stress/plastic strain curve. In order to derive the stress or strain components from a calculated value of the effective stress or strain, the flow behaviour must be characterised.

The flow law is

$$d\epsilon_{pi} = \frac{d\bar{\epsilon}_p}{c} \frac{\partial G}{\partial \sigma_i} \quad (11)$$

and relates components i of increments of plastic strain $d\epsilon_p$ to components of stress σ_i . G and c are functions of stress components and a flow parameter ψ . The magnitude of the parameter ψ is determined by the direction of plastic flow such that $\tan \psi$ is the ratio of the change in hydrostatic plastic strain to the change in deviatoric plastic strain. Thus

$$\tan \psi = \frac{3(1-2v_p - 2v_p' \bar{\epsilon}_p)}{2(1+v_p + v_p' \bar{\epsilon}_p)} \quad (12)$$

where v_p is the plastic component of the lateral contraction ratio and v_p' is the derivative with respect to the effective plastic strain. Equation (12) allows for a variation in v_p , and hence ψ , with strain. If this is insignificant, a constant value for ψ is given by

$$\tan \psi = \frac{3(1-2\nu_p)}{2(1+\nu_p)} \quad (13)$$

If $\psi=0$, then plastic flow takes place under constant volume.

3.3 SUMMARY OF SECTION 3

Each of the elastic-plastic models requires the following information:

- the Young's modulus E and Poisson's ratio ν_e - needed to characterise linear behaviour and to deduce the plastic component of strain ϵ_p from a total strain value
- the tensile yield stress as a function of plastic strain $\sigma_T(\epsilon_p)$ - needed to characterise strain hardening
- the hydrostatic-stress sensitivity parameter λ or β - needed in the yield criterion as a measure of the sensitivity of yield behaviour to a hydrostatic component of stress
- the flow parameter ψ - needed by the flow rule and deduced from measurements of lateral contraction ratio $\nu_p(\epsilon_p)$ as a function of plastic strain.

4 EXTENSION TO RATE-DEPENDENT BEHAVIOUR

For materials that have time and rate-dependent properties, the parameters introduced in section 3 and summarised in 3.3 will vary with the rate of application of strain. At present, most of the FE packages that include rate dependence in their elastic-plastic models only accommodate dependence of the yield stress σ_T . The strain and strain-rate dependence of σ_T is often handled by means of separable functions thus

$$\sigma_T(\epsilon_p, \dot{\epsilon}_p) = f(\epsilon_p) g(\dot{\epsilon}_p) \quad (14)$$

Equation (14) implies that changes in the rate of straining of a material do not influence the strain dependence and vice versa. Some examples of the forms assumed for $g(\dot{\epsilon}_p)$ are

$$g(\dot{\epsilon}_p) = \dot{\epsilon}_p^m \quad (\text{power law}) \quad (15)$$

$$g(\dot{\epsilon}_p) = \left(1 + \ln \frac{\dot{\epsilon}_p}{\dot{\epsilon}_o} \right) \quad (\text{Eyring}) \quad (16)$$

$$\text{and } g(\dot{\epsilon}_p) = \left(1 + \left(\frac{\dot{\epsilon}_p}{C} \right)^{1/p} \right) \quad (\text{Cowper-Symonds}) \quad (17)$$

Here, $\dot{\epsilon}_0$ is a reference strain rate at which $\sigma_T(\epsilon)$ is determined.

Using these functions, only one or two parameters are needed to characterise the rate-dependence of yield stress. For materials or temperature ranges where the use of separable functions is of limited accuracy, an alternative approach, adopted by some packages, is to model strain dependence by a suitable parametric function with rate dependent parameters. Then $\sigma_T(\epsilon_p, \dot{\epsilon}_p)$ might take the form

$$\sigma_T(\epsilon_p, \dot{\epsilon}_p) = a(\dot{\epsilon}_p) + b(\dot{\epsilon}_p) \epsilon_p^{n(\dot{\epsilon}_p)} \quad (18)$$

A larger number of parameters is then required to characterise rate dependence but this is not excessive if a bilinear representation of σ_T ($n=1$ in equation (18)) is adopted.

5 EXPERIMENTAL DETERMINATION OF DATA

A summary of the parameters required by elastic-plastic models is given in section 3.3. Test methods for generating the experimental data from which these parameters can be derived will now be considered.

5.1 E, ν_e

Young's modulus and Poisson's ratio can be obtained from a tensile test with suitable extensometry carried out under constant strain rate and at small strains where behaviour is linear. The rate dependence of these properties can be established by performing tests over a wide range of strain rate. For highly viscoelastic materials, stress/strain plots are curved even at small strains where behaviour is linear. Rate dependence is then best described using stress relaxation tests (7).

5.2 $\sigma_T(\epsilon_p)$

The function $\sigma_T(\epsilon_p)$ is derived from tensile test data obtained beyond the linear range of strain and up to failure. For most structural plastics, this means tensile strains above 0.01

to 0.5 or more. The higher levels of strain are generally beyond the range of contacting extensometers, and alternative devices, such as a video extensometer, will be necessary. Furthermore at strains above 0.1, many plastics exhibit strain softening or necking which manifests itself as a local reduction in the cross-sectional area of the specimen. Recorded stresses and strains are then usually only average values and results are not suitable for stress analysis. Where components are subjected to impact, most of the energy dissipated by plastics takes place at strains above 0.1 and, for valid stress analyses, it is necessary to generate true stress/true strain data. The derivation of true stress requires measurement of cross-section dimensions at longitudinal strains greater than 0.05. If strain softening is encountered, local longitudinal strain measurements will also be necessary in the region of the neck. Certain video extensometers are claimed to be capable of such measurements. Furthermore, at strains above 0.05, it is more accurate to use true strain data rather than engineering strain values (see Appendix 1).

Measurements of the strain rate dependence of $\sigma_T(\epsilon_p)$ are also needed. These should span as wide a range of strain rate as possible. At rates significantly above 0.1 s^{-1} , commercial extensometers are unreliable. Furthermore strain gauges are inaccurate, especially for unreinforced plastics, and their strain range is insufficient.

5.3 λ, β

The parameters λ and β are related by equations (6). Their determination involves the measurement of yield behaviour under two different states of stress. Uniaxial tension is an obvious choice for one of these states. If, in addition, stress/strain curves under uniaxial compression are obtained, then λ can be derived from the ratio of compressive to tensile stresses at the same effective strain. The measurement of stress/strain data under uniaxial compression is made difficult by the instability of the test specimen, and reliable data cannot be obtained without extreme precautions. A shear test (8) has been considered as an alternative test for the determination of λ and has been employed to generate stress/strain data for a number of plastics at strains up to and beyond the region where the stress level reaches a maximum.

The analysis of data from tensile and shear tests will be illustrated with reference to the linear Drucker-Prager model. The yield criterion for this model is given by equation (5). If a yield stress in a pure shear test is σ_s at some effective plastic strain then, from (5) and (6)

$$\sqrt{3}\sigma_s = d$$

(19)

and
$$\frac{\sqrt{3}\sigma_s}{\sigma_T} = \frac{2\lambda}{\lambda+1}$$

where σ_T and d refer to the same effective plastic strain. We can now define a new ratio of yield stresses given by the parameter μ where, from equations (19)

$$\mu = \frac{\sqrt{3}\sigma_s}{\sigma_T} = \frac{2\lambda}{\lambda+1} \quad (20)$$

The parameter λ can now be replaced in the yield criterion, and equations (3) and (5) become

$$\sigma_T = \frac{\sqrt{3}}{\mu} J_{2D}^{\frac{1}{2}} + \frac{\mu-1}{\mu} J_1 \quad (21)$$

and

$$d = \mu\sigma_T = q - p\tan\beta \quad (22)$$

where

$$\tan\beta = 3(\mu-1) \quad (23)$$

Von-Mises yielding is obtained for $\mu=1$.

So the parameters λ or β can be derived from tensile and shear test data through a determination of μ .

The most simple and direct method for determining a value for μ is from the maximum stresses on stress/strain curves obtained in tension and shear. These yield stresses clearly lie on the same yield surface which is the outermost one corresponding to maximum strain hardening. This simple derivation is not possible if the plastic does not exhibit a clear stress maximum, ie if it continues to strain harden. Under these circumstances an alternative procedure is necessary to identify equivalent stresses on tensile and shear curves. This procedure is based on the use of the effective plastic strain and is explained in Appendix 2. This procedure also allows values for μ to be derived at any plastic strain value thereby demonstrating if μ varies with strain and whether the single value obtained from stress maxima is a realistic value. This exercise is primarily academic since most FE packages cannot accept a strain dependence of hydrostatic stress sensitivity.

The hydrostatic stress sensitivity parameter will, in general, depend upon strain rate. The range of strain rate that can be covered by shear measurements is not as wide as potentially achievable in tension. The acquisition of values for μ corresponding to high rates is achievable, with probably sufficient accuracy, by extrapolation.

5.4 ψ, v_p

Determination of the flow parameter ψ requires measurements of longitudinal and lateral true strain throughout the non-linear region of a tensile test. These strain values are composed of elastic and plastic components which can be determined from a knowledge of E and ν . The plastic component v_p of the lateral contraction ratio can then be deduced from the ratio of the transverse plastic strain to the longitudinal plastic strain (see Appendix 1). Any variation of v_p with longitudinal plastic strain (the effective plastic strain) will be apparent. A representative value will need to be selected and used in equation (13) to determine ψ . Although equation (12) could be used to derive a more accurate value for ψ , FE packages cannot accommodate a strain dependence of this parameter.

6 CONCLUSIONS

A finite element analysis of the response of plastics to an impact loading requires the use of a materials model that is capable of describing the behaviour of the material under high strains where the relationship between stress and strain is non-linear. Since plastics are viscoelastic, the dependence of properties on the rate of loading also needs to be included.

The most widely used models for impact analysis are elastic-plastic models that have been extended so that some of the parameters are rate dependent. These plasticity models have been developed for metals, and limitations in their applicability to plastics are not clear.

The yield criteria in models that are relevant to plastics need to include the sensitivity of yield behaviour to the hydrostatic component of stress. Unless the stress state becomes large and triaxial there is probably little difference between the accuracies of the various criteria. In general, the same material parameters need to be determined for these criteria.

The material parameters are:

$$E, \nu, \sigma_T(\epsilon_p), v_p, \lambda \text{ and } \psi$$

Stress/strain curves under two different states of stress are required to determine λ . Tensile and shear tests have been recommended here. Measurements of lateral and longitudinal strains are needed throughout the tensile tests. If the specimen necks, these measurements must be made at local regions of the specimen.

An approximate analysis is obtained using $\lambda = \mu = 1$, $\nu_p = 0.5$ and $\psi = 0$. The significance of errors incurred by the use of these approximations for different loading situations is unknown.

Reliable methods for measuring tensile strain at strain rates above 1 s^{-1} need to be developed.

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APPENDIX 1 Nomenclature and basic relationships

E	Young's modulus.
σ_i	i th principal stress component.
σ_T	tensile yield stress.
σ_s	shear yield stress.
$\epsilon, \epsilon_e, \epsilon_p$	engineering tensile strain, elastic component, plastic component.
ϵ_{pi}	i th principal engineering plastic strain component.
$\epsilon_t, \epsilon_{te}, \epsilon_{tp}$	transverse engineering tensile strain, elastic component, plastic component.
$\gamma, \gamma_e, \gamma_p$	shear strain, elastic component, plastic component.
ν, ν_e, ν_p	lateral contraction ratio, elastic component (= Poisson's ratio), plastic component.
$\ln(1+\epsilon)$	true strain.
$\bar{\sigma}$	effective stress.
$\bar{\epsilon}_p$	effective plastic strain.

In terms of true strains

$$\ln(1+\epsilon) = \ln(1+\epsilon_e) + \ln(1+\epsilon_p) \quad A1$$

$$\nu = - \frac{\ln(1+\epsilon_t)}{\ln(1+\epsilon)} \quad A2$$

$$\nu_p = - \left[\frac{\ln(1+\epsilon_t) - \ln(1+\epsilon_{te})}{\ln(1+\epsilon_p)} \right] \quad A3$$

and

$$\nu_e = - \frac{\epsilon_{te}}{\epsilon_e} \quad A4$$

where ϵ_t and ϵ_{te} are negative.

Expression for $\bar{\sigma}$ and $\bar{\epsilon}_p$ depend on the yield criterion. If the Drucker-Prager criterion is used these become

$$\bar{\sigma} = \frac{\sqrt{3}(\lambda+1)}{2\lambda} J_{2D}^{\frac{1}{2}} + \frac{(\lambda-1)}{2\lambda} J_1 \quad A5$$

$$= \frac{\sqrt{3}}{\mu} J_{2D}^{\frac{1}{2}} + \frac{(\mu-1)}{\mu} J_1 \quad A6$$

$$\text{and } \bar{\epsilon}_p = \frac{\sqrt{3}(\lambda+1)}{2\lambda(1+\nu_p)} I_{2D}^{\frac{1}{2}} + \frac{(\lambda-1)}{2\lambda(1-2\nu_p)} I_1 \quad A7$$

$$= \frac{\sqrt{3}}{\mu} \frac{I_{2D}^{\frac{1}{2}}}{(1+\nu_p)} + \frac{(\mu-1)}{\mu} \frac{I_1}{(1-2\nu_p)} \quad A8$$

where

$$I_1 = \epsilon_{p1} + \epsilon_{p2} + \epsilon_{p3}$$

$$\text{and } I_{2D} = \frac{1}{6} [(\epsilon_{p1} - \epsilon_{p2})^2 + (\epsilon_{p2} - \epsilon_{p3})^2 + (\epsilon_{p3} - \epsilon_{p1})^2]$$

At engineering strain levels above 0.05, ϵ_p should be replaced by the true strain $\ln(1 + \epsilon_p)$.

APPENDIX 2 Experimental determination of μ

This appendix explains the data analysis that can be used to determine the parameter μ as a function of plastic strain. Experimental stress/strain curves in tension and shear are needed.

From the total strain measurements in tension and shear, the plastic components ϵ_p and γ_p are derived and plots of σ_T against ϵ_p and σ_s against γ_p generated. Examples are given in figure A1. For any point σ_T, ϵ_p on the tensile plot, a secant modulus E_s can be defined as

$$E_s = \frac{\sigma_T}{\epsilon_p} \quad A9$$

Similarly from the shear plot, a secant shear modulus is

$$G_s = \frac{\sigma_s}{\gamma_p} \quad A10$$

From equations A4 and A6 or A5 and A7

$$E_s = \frac{\sigma_T}{\epsilon_p} = \frac{\bar{\sigma}}{\bar{\epsilon}_p} \quad A11$$

and

$$G_s = \frac{\sigma_s}{\gamma_p} = \frac{\bar{\sigma}}{\bar{\epsilon}_p} \frac{1}{2(1+\nu_p)} = \frac{E_s}{2(1+\nu_p)} \quad A12$$

where, if ν_p is not constant, it is the value at $\bar{\epsilon}_p$.

Thus for any point σ_T, ϵ_p on the tensile curve G_s can be calculated using A12 and the equivalent point σ_s and γ_p can be determined on the shear curve. σ_T and σ_s are then equivalent stresses, and

$$\mu = \frac{\sqrt{3}\sigma_s}{\sigma_T} \text{ at } \epsilon_p = \bar{\epsilon}_p$$

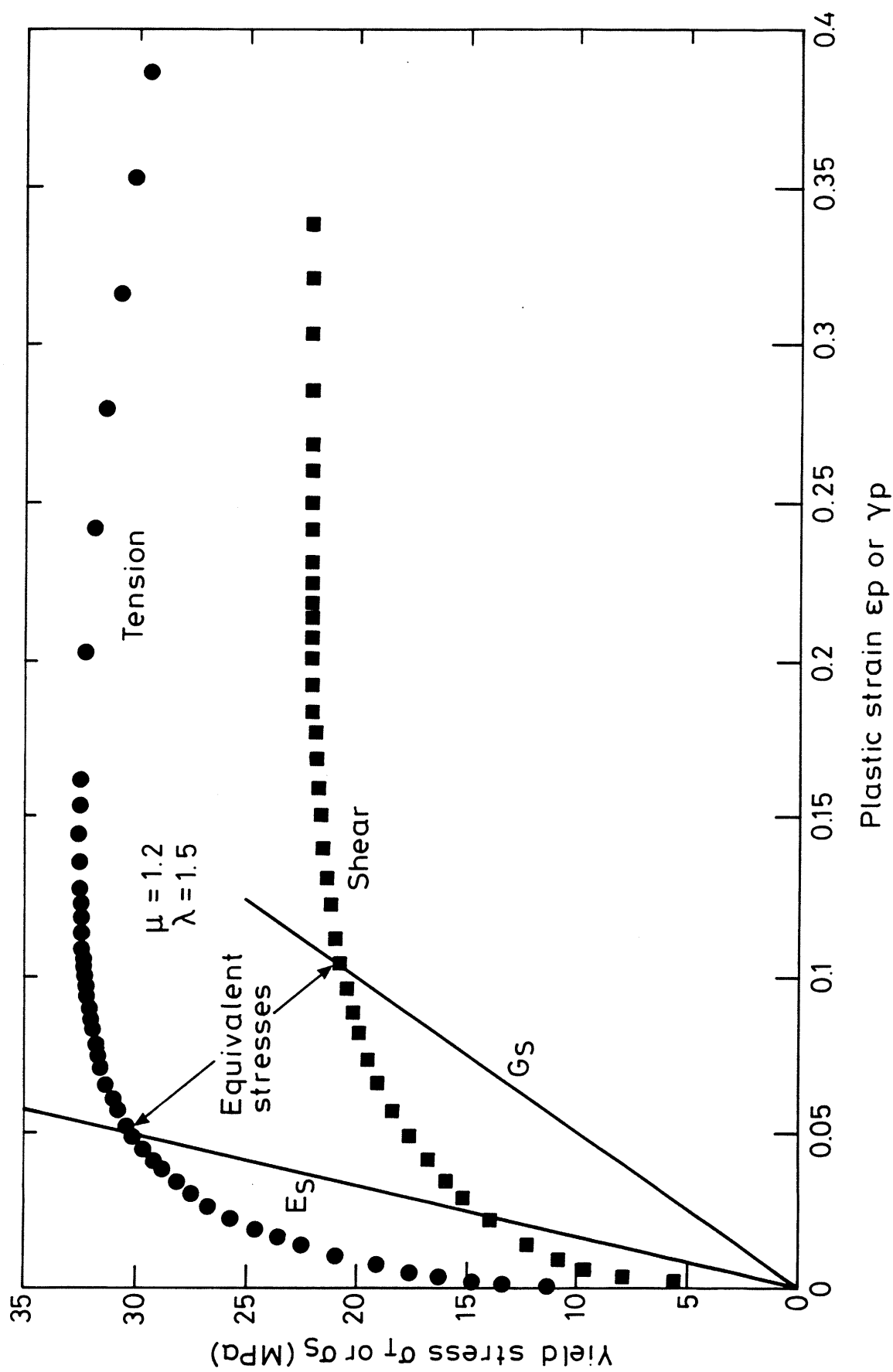


Figure A1 Plots of tensile yield stress σ_T against tensile plastic strain ϵ_p and shear yield stress σ_S against shear plastic strain γ_p for polypropylene

