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STRESS TRANSFER MECHANICS FOR PLY CRACKS IN GENERAL SYMMETRIC LAMINATES

by

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SUMMARY

The report provides the details of the development of a new stress transfer model for predicting the stress and displacement distributions in a general symmetric laminate having a uniform distribution of cracks in some or all of the 90° plies. The laminate can be subject to any combination of in-plane biaxial and shear loading, and the analysis takes full account of the effects of residual stresses arising from thermal expansion and moisture effects, assuming that the temperature and moisture content are uniform throughout the laminate. The method of predicting the effective hygro-thermo-elastic constants of the cracked laminate is described, and it is shown by considering ply crack closure that various inter-relationships must be satisfied and this has been verified by specific applications of the theory. Preliminary predictions of the dependence of the hygro-thermo-elastic constants on crack density have been made for quasi-isotropic [-45/45/0/90]_s carbon fibre and [30/90]_s glass fibre epoxy laminates.

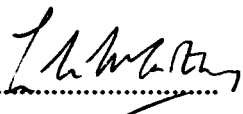
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1. Introduction

The prediction of the initiation and progressive growth of ply cracks in laminated composites has received a great deal of attention in the literature whose detailed review is beyond the scope of this paper. Much work has concerned the prediction of ply cracking in cross-ply laminates subject to uniaxial loading using either shear-lag approaches (e.g. [1]) or variational techniques (e.g. [2-8]). Complete solution techniques that predict both stress and displacement distributions in a cracked laminate have been developed by the author for multiple-ply cross-ply and simple angle-ply laminates [9-14] subject to in-plane biaxial loading. The analytical methods and associated software have now been thoroughly tested and the methodology extended to the practically important case of general symmetric laminates that is to be considered in this report.

The objective is to provide the details of the theoretical modelling that has been used to develop a reliable technique for predicting stress transfer between the neighbouring plies of a general symmetric laminate that may have uniformly spaced ply cracks in some of the plies having a 90° orientation with respect to the principal loading axial direction. Generalised plane strain conditions are assumed that are expected to be valid for those regions in any laminate subject to general in-plane deformation that are not affected by edge stress transfer effects. Generalised plane strain deformation, while reducing the problem to one of essentially two dimensions, does take account of the effects of general in-plane loading of the laminate characterised by biaxial in-plane loading combined with in-plane shear loading. The methodology assumes that there is perfect bonding between the plies of the laminate, and the analysis is developed so that account can be taken of the effects of both uniform temperature and moisture changes. The technique is basically analytical, but because of the resulting complexity, the analysis must be handled numerically when making predictions of the behaviour of the laminate. In applications of the analysis, ply refinement techniques may be used where each ply of the laminate is subdivided into layers having the same properties in order that through-thickness variations of the stress and displacement components can be taken into account. Such through-thickness variations are of importance in the neighbourhoods of ply cracks.

The methodology to be developed is based essentially on a single assumption where the through-thickness shear stresses are assumed to be linear functions of the through-thickness coordinate in each layer of all the plies. For this representation unknown functions of the axial coordinate are introduced that represent the shear stresses on the interfaces between each layer. A consequence of this assumption is that the axial stress in each layer is independent of the through-thickness coordinate. The equilibrium equations are used to determine the corresponding distributions of the remaining relevant stress components. Three of the six independent stress/strain relations are then used to determine the displacement field that corresponds exactly to the stress field. The remaining three in-plane stress/strain relations are applied, on averaging through the thickness of each layer, to derive a set of simultaneous fourth order ordinary differential equations that determine the unknown functions of the stress and displacement field representation. It has been shown (unpublished work) that the averaging procedure adopted ensures that the stress and displacement field assumed leads to a stationary value for the Reissner energy functional [15] that has been applied to laminates by Pagano [16]. Thus the solution developed here would be wholly consistent with solutions obtained using such a variational technique. The representation is thus one of high quality and can be expected, on using ply refinement techniques, to lead to solutions for the stress and displacement fields that are almost exact. The numerical solution method is such that the governing differential equations and associated boundary conditions can be checked for accuracy.

It is emphasised that the methodology provides the distribution of both the stress field and the displacement field, unlike variational techniques based on complementary energy that

do not lead to complete solutions. It is thus possible to predict all the effective hygro-thermo-elastic constants for a cracked laminate. Crack closure arguments will be used to derive inter-relationships between these constants that must always be satisfied. The satisfaction of these inter-relationships can be used as a measure of the quality of a stress transfer model, and of the associated numerical analysis.

2. Geometry and basic field equations

The problem under consideration concerns the in-plane deformation of a symmetric multi-layered laminate constructed of $2N+2$ perfectly bonded layers which can have any combination of orientations provided that laminate symmetry is preserved. As symmetry about the mid-plane of the laminate is assumed, it is necessary to consider only the right hand set of $N+1$ layers as shown in Fig.1. A global set of rectangular Cartesian co-ordinates are chosen having the origin at the centre of the laminate as shown in the figure. The y -direction defines the longitudinal or axial direction, the z -direction defines the transverse direction and the x -direction defines the through-thickness direction. The locations of the N interfaces are specified by $x = x_i$, $i = 1...N$. The mid-plane of the laminate is specified by $x = x_0 = 0$ and the external surface by $x = x_{N+1} = h$ where $2h$ is the total thickness of the laminate. The thickness of the i^{th} layer is denoted by $h_i = x_i - x_{i-1}$. The orientation of the i^{th} layer is specified by the angle ϕ_i (measured clockwise) between the y -axis and the fibre direction of this layer. In addition to the global coordinates (x, y, z) , sets of local rectangular Cartesian coordinates (x'_i, y'_i, z'_i) , $i = 1...N+1$, are introduced in each of the layers so that the fibres in the i^{th} layer are parallel to the y'_i -axis. The z'_i -axis is directed in the in-plane transverse direction of the i^{th} layer and the x'_i -axis is in the through-thickness direction. Both local and global coordinates share the same origin which is located on the plane of symmetry of the laminate.

The region to be considered occupies the region $|x| \leq h$, $|y| \leq L$, $|z| \leq W$ of the laminate, and is such that the faces of the laminate are stress-free. Ply cracks in some or all of the 90° plies are assumed to form only on the planes $y = \pm L$ indicating that the laminate must be such that the orientation of fibres in at least one set of plies is aligned in the z -direction. This assumption is not a limitation as general in-plane loading conditions are considered so that if cracks form in another single orientation the laminate can be rotated so that the crack planes are parallel to the z -axis and the applied stresses transformed to values appropriate for the new orientation. Stress, strain and displacement components, and material properties associated with the i^{th} layer are denoted by a superscript or subscript i .

The following equilibrium equations, for stress components σ_{xx} , σ_{xy} , etc. referred to global coordinates, must be satisfied for $i = 1...N+1$

$$\frac{\partial \sigma_{xx}^i}{\partial x} + \frac{\partial \sigma_{xy}^i}{\partial y} + \frac{\partial \sigma_{xz}^i}{\partial z} = 0, \quad (1)$$

$$\frac{\partial \sigma_{xy}^i}{\partial x} + \frac{\partial \sigma_{yy}^i}{\partial y} + \frac{\partial \sigma_{yz}^i}{\partial z} = 0, \quad (2)$$

$$\frac{\partial \sigma_{xz}^i}{\partial x} + \frac{\partial \sigma_{yz}^i}{\partial y} + \frac{\partial \sigma_{zz}^i}{\partial z} = 0. \quad (3)$$

The stress-strain relations referred to global coordinates are given, for $i = 1...N+1$, by

$$\epsilon_{xx}^i \equiv \frac{\partial u_i}{\partial x} = g_{11}^i \sigma_{xx}^i + g_{12}^i \sigma_{yy}^i + g_{13}^i \sigma_{zz}^i + g_{14}^i \sigma_{yz}^i + \alpha_1^i \Delta T + \beta_1^i \Delta c, \quad (4)$$

$$\epsilon_{yy}^i \equiv \frac{\partial v_i}{\partial y} = g_{12}^i \sigma_{xx}^i + g_{22}^i \sigma_{yy}^i + g_{23}^i \sigma_{zz}^i + g_{24}^i \sigma_{yz}^i + \alpha_2^i \Delta T + \beta_2^i \Delta c, \quad (5)$$

$$\epsilon_{zz}^i \equiv \frac{\partial w_i}{\partial z} = g_{13}^i \sigma_{xx}^i + g_{23}^i \sigma_{yy}^i + g_{33}^i \sigma_{zz}^i + g_{34}^i \sigma_{yz}^i + \alpha_3^i \Delta T + \beta_3^i \Delta c, \quad (6)$$

$$2\epsilon_{yz}^i \equiv \frac{\partial v_i}{\partial z} + \frac{\partial w_i}{\partial y} = g_{14}^i \sigma_{xx}^i + g_{24}^i \sigma_{yy}^i + g_{34}^i \sigma_{zz}^i + g_{44}^i \sigma_{yz}^i + \alpha_4^i \Delta T + \beta_4^i \Delta c, \quad (7)$$

$$\epsilon_{xy}^i \equiv \frac{1}{2} \left(\frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial x} \right) = a_{11}^i \sigma_{xy}^i + a_{12}^i \sigma_{xz}^i, \quad (8)$$

$$\epsilon_{xz}^i \equiv \frac{1}{2} \left(\frac{\partial u_i}{\partial z} + \frac{\partial w_i}{\partial x} \right) = a_{12}^i \sigma_{xy}^i + a_{22}^i \sigma_{xz}^i, \quad (9)$$

where ΔT and Δc are the uniform temperature and moisture content differences, defined respectively by $\Delta T = T - T_0$ and $\Delta c = c - c_0$ in which T and c are the current temperature and moisture content of the material, and T_0 and c_0 are the temperature and moisture content for which the strain is zero and the material is everywhere stress-free, with no internal or imposed external stresses. The parameters u_i , v_i and w_i denote the displacement components for the i^{th} layer in the x , y and z directions respectively, and ϵ_{xx} , ϵ_{xy} , etc. denote the infinitesimal strain components.

By using the procedure defined in Appendix A, and letting $m_i = \cos\phi_i$ and $n_i = \sin\phi_i$, it can be shown that the coefficients in the stress-strain relations (4-9) are related to the hygro-thermo-elastic constants of the layers as follows, for values $i = 1...N+1$:

$$g_{11}^i = \frac{1}{E_t^i},$$

$$g_{12}^i = -n_i^2 \frac{v_t^i}{E_t^i} - m_i^2 \frac{v_a^i}{E_a^i},$$

$$g_{13}^i = -m_i^2 \frac{v_t^i}{E_t^i} - n_i^2 \frac{v_a^i}{E_a^i}$$

$$g_{14}^i = 2m_i n_i \frac{v_t^i}{E_T^i} \frac{v_a^i}{E_A^i} \tag{13}$$

$$g_{22}^i = m_i^2 n_i^2 \frac{1}{\mu_A^i} \frac{2v_A^i}{E_A^i} \frac{n_i}{E_T^i} \frac{m_i}{E^i}$$

$$g_{23}^i = m_i^2 n_i \frac{1}{E^i} \frac{1}{E_T^i} \frac{1}{\mu_A^i} m_i \frac{v_A^i}{E^i} n_i \frac{v_A^i}{E^i}$$

$$g_{34}^i = m_i n_i (m_i + n_i^2) \frac{1}{\mu_A^i} \frac{2v_A^i}{E^i} \frac{2m_i}{E_T^i} \frac{2n_i^2}{E_T^i} \tag{6}$$

$$g_{33}^i = m_i^2 n_i^2 \frac{1}{\mu_A^i} \frac{2v_A^i}{E} \frac{m_i}{E_T^i} \frac{n_i}{E^i}$$

$$g_{24}^i = m_i n_i (m_i + n_i^2) \frac{1}{\mu_A^i} \frac{2v_A^i}{E_A^i} \frac{2m_i}{E_T^i} \frac{2n_i^2}{E_A^i} \tag{18}$$

$$g_{44}^i = \frac{(m_i + n_i^2)}{\mu^i} m_i^2 n_i^2 \frac{1}{E^i} \frac{1}{E_T^i} \frac{2v_A^i}{E^i} \tag{19}$$

$$\alpha_1^i = \alpha_t^i \quad \beta_1^i = \beta_t^i \tag{20}$$

$$\alpha_2^i = m_i^2 \alpha_A^i + n_i^2 \alpha_T^i \quad \beta_2^i = m_i^2 \beta_A^i + n_i^2 \beta_T^i$$

$$\alpha_3^i = m_i^2 \alpha_T^i + n_i^2 \alpha_A^i \quad \beta_3^i = m_i^2 \beta_T^i + n_i^2 \beta_A^i \tag{22}$$

$$\alpha_4^i = 2m_1n_1(\alpha_T^i - \alpha_A^i), \quad \beta_4^i = 2m_1n_1(\beta_T^i - \beta_A^i), \quad (23)$$

$$a_{11}^i = \frac{1}{2} \left(\frac{m_1^2}{\mu_a^i} + \frac{n_1^2}{\mu_t^i} \right) \quad (24)$$

$$a_{12}^i = \frac{1}{2} m_1n_1 \left(\frac{1}{\mu_t^i} - \frac{1}{\mu_a^i} \right) \quad (25)$$

$$a_{22}^i = \frac{1}{2} \left(\frac{m_1^2}{\mu_t^i} + \frac{n_1^2}{\mu_a^i} \right) \quad (26)$$

In the relations (10-26) the parameters E , ν , μ , α and β denote the Young's modulus, Poisson's ratio, shear modulus, thermal expansion coefficient and moisture expansion coefficient respectively. The hygro-thermo-elastic constants E , ν , μ , α and β are allowed to have different values in each ply of the laminate. Each ply is assumed to be orthotropic so that fifteen independent hygro-thermo-elastic constants are required to characterise linear behaviour. The upper case subscripts A and T, attached to axial and transverse hygro-thermo-elastic constants appearing in (A7-A12), denote that they refer to in-plane stresses and deformation while the corresponding lower case subscripts denote hygro-thermo-elastic constants that involve out-of-plane stresses and deformations.

3. Generalised in-plane strain conditions

The fundamental assumption for generalised in-plane strain conditions is that the displacement field is of the following form

$$u = u(x, y), \quad v = v(x, y), \quad w = f(x, y) + \epsilon_c^* z, \quad (27)$$

where ϵ_c^* is the transverse strain of the cracked laminate that has the same value in all layers. This representation is valid only if ply cracks form in planes normal to the y -direction, and only if well away from the edges of the laminate. From (6) and (27)₃ it follows that the transverse stress is given by

$$\sigma_{zz}^i = \frac{1}{g_{33}^i} \left[\epsilon_c^* - g_{13}^i \sigma_{xx}^i - g_{23}^i \sigma_{yy}^i - g_{34}^i \sigma_{yz}^i - \alpha_3^i \Delta T - \beta_3^i \Delta c \right]. \quad (28)$$

The stress-strain relations (4), (5) and (7) may then be written in the following reduced form

$$\epsilon_{xx}^i \equiv \frac{\partial u_i}{\partial x} = \bar{g}_{11}^i \sigma_{xx}^i + \bar{g}_{12}^i \sigma_{yy}^i + \bar{g}_{14}^i \sigma_{yz}^i + \bar{\alpha}_1^i \Delta T + \bar{\beta}_1^i \Delta c + H_1^i \epsilon_c^*, \quad (29)$$

$$\epsilon_{yy}^i \equiv \frac{\partial v_i}{\partial y} = \bar{g}_{12}^i \sigma_{xx}^i + \bar{g}_{22}^i \sigma_{yy}^i + \bar{g}_{24}^i \sigma_{yz}^i + \bar{\alpha}_2^i \Delta T + \bar{\beta}_2^i \Delta c + H_2^i \epsilon_c^*, \quad (30)$$

$$2\epsilon_{yz}^i \equiv \frac{\partial v_i}{\partial z} + \frac{\partial w_i}{\partial y} = \bar{g}_{14}^i \sigma_{xx}^i + \bar{g}_{24}^i \sigma_{yy}^i + \bar{g}_{44}^i \sigma_{yz}^i + \bar{\alpha}_4^i \Delta T + \bar{\beta}_4^i \Delta c + H_3^i \epsilon_c^*, \quad (31)$$

where

$$\begin{aligned} H_1^i &= \frac{g_{13}^i}{g_{33}^i}, & H_2^i &= \frac{g_{23}^i}{g_{33}^i}, & H_3^i &= \frac{g_{34}^i}{g_{33}^i}, \\ \bar{g}_{11}^i &= g_{11}^i - \frac{(g_{13}^i)^2}{g_{33}^i}, & \bar{g}_{12}^i &= g_{12}^i - \frac{g_{13}^i g_{23}^i}{g_{33}^i}, & \bar{g}_{14}^i &= g_{14}^i - \frac{g_{13}^i g_{34}^i}{g_{33}^i}, \\ \bar{g}_{22}^i &= g_{22}^i - \frac{(g_{23}^i)^2}{g_{33}^i}, & \bar{g}_{24}^i &= g_{24}^i - \frac{g_{23}^i g_{34}^i}{g_{33}^i}, & \bar{g}_{44}^i &= g_{44}^i - \frac{(g_{34}^i)^2}{g_{33}^i}, \\ \bar{\alpha}_1^i &= \alpha_1^i - \frac{g_{13}^i}{g_{33}^i} \alpha_3^i, & \bar{\alpha}_2^i &= \alpha_2^i - \frac{g_{23}^i}{g_{33}^i} \alpha_3^i, & \bar{\alpha}_4^i &= \alpha_4^i - \frac{g_{34}^i}{g_{33}^i} \alpha_3^i, \\ \bar{\beta}_1^i &= \beta_1^i - \frac{g_{13}^i}{g_{33}^i} \beta_3^i, & \bar{\beta}_2^i &= \beta_2^i - \frac{g_{23}^i}{g_{33}^i} \beta_3^i, & \bar{\beta}_4^i &= \beta_4^i - \frac{g_{34}^i}{g_{33}^i} \beta_3^i. \end{aligned} \quad (32)$$

4. Solution for undamaged laminates

The external faces of the laminate are assumed to be stress-free so that for all points in the laminate away from the laminate edges

$$\sigma_{xx}^i \equiv 0, \quad \sigma_{xy}^i \equiv 0, \quad \sigma_{xz}^i \equiv 0, \quad i = 1 \dots N+1. \quad (33)$$

It then follows from (8), (9) and (33) that

$$\epsilon_{xy}^i \equiv \epsilon_{xz}^i \equiv 0, \quad (34)$$

and from (29-31) and (33), on setting $\epsilon_c^* = \epsilon^*$, that

$$\epsilon_i = \bar{g}_{12}^i \sigma_i + \bar{g}_{14}^i \tau_i + \bar{\alpha}_1^i \Delta T + \bar{\beta}_1^i \Delta c + H_1^i \epsilon^*, \quad (35)$$

$$\varepsilon = \bar{g}_{22}^i \sigma_i + \bar{g}_{24}^i \tau_i + \bar{\alpha}_2^i \Delta T + \bar{\beta}_2^i \Delta c + H_2^i \varepsilon^* , \quad (36)$$

$$\gamma = \bar{g}_{24}^i \sigma_i + \bar{g}_{44}^i \tau_i + \bar{\alpha}_4^i \Delta T + \bar{\beta}_4^i \Delta c + H_3^i \varepsilon^* , \quad (37)$$

where ε , ε^* , and γ are respectively the uniform axial strain, transverse strain and in-plane engineering shear strain in an undamaged laminate, where $\bar{\varepsilon}_i$ is the uniform out-of-plane strain in the i^{th} layer, and where σ_i and τ_i are respectively the uniform axial and in-plane shear stresses in the i^{th} layer of an undamaged laminate. From (28) the corresponding transverse in-plane stress component in the i^{th} layer is given by

$$\sigma_{zz}^i = \sigma_i^* = \frac{1}{\bar{g}_{33}^i} \left[\varepsilon^* - \bar{g}_{23}^i \sigma_i - \bar{g}_{34}^i \tau_i - \bar{\alpha}_3^i \Delta T - \bar{\beta}_3^i \Delta c \right] . \quad (38)$$

The relations (36) and (37) may be rearranged leading to the following expressions for the axial and transverse stresses in the i^{th} layer, for $i = 1 \dots N+1$

$$\begin{aligned} \sigma_{yy}^i = \sigma_i = \frac{1}{\bar{g}_{22}^i \bar{g}_{44}^i - (\bar{g}_{24}^i)^2} & \left[\bar{g}_{44}^i \varepsilon - \bar{g}_{24}^i \gamma + (H_3^i \bar{g}_{24}^i - H_2^i \bar{g}_{44}^i) \varepsilon^* \right. \\ & \left. + (\bar{g}_{24}^i \bar{\alpha}_4^i - \bar{g}_{44}^i \bar{\alpha}_2^i) \Delta T + (\bar{g}_{24}^i \bar{\beta}_4^i - \bar{g}_{44}^i \bar{\beta}_2^i) \Delta c \right] \end{aligned} \quad (39)$$

$$\begin{aligned} \sigma_{yz}^i = \tau_i = \frac{1}{\bar{g}_{22}^i \bar{g}_{44}^i - (\bar{g}_{24}^i)^2} & \left[-\bar{g}_{24}^i \varepsilon + \bar{g}_{22}^i \gamma + (H_2^i \bar{g}_{24}^i - H_3^i \bar{g}_{22}^i) \varepsilon^* \right. \\ & \left. + (\bar{g}_{24}^i \bar{\alpha}_2^i - \bar{g}_{22}^i \bar{\alpha}_4^i) \Delta T + (\bar{g}_{24}^i \bar{\beta}_2^i - \bar{g}_{22}^i \bar{\beta}_4^i) \Delta c \right] \end{aligned} \quad (40)$$

The effective applied axial, transverse and shear stresses are given respectively by

$$\sigma = \frac{1}{h} \sum_{i=1}^{N+1} h_i \sigma_i = A \varepsilon + B \varepsilon^* + D \gamma + P \Delta T + U \Delta c , \quad (41)$$

$$\sigma^* = \frac{1}{h} \sum_{i=1}^{N+1} h_i \sigma_i^* = B \varepsilon + C \varepsilon^* + E \gamma + Q \Delta T + V \Delta c , \quad (42)$$

$$\tau = \frac{1}{h} \sum_{i=1}^{N+1} h_i \tau_i = X \varepsilon + Y \varepsilon^* + F \gamma + R \Delta T + W \Delta c , \quad (43)$$

where

$$A = \frac{1}{h} \sum_{i=1}^{N+1} h_i \frac{\bar{g}_{44}^i}{\bar{g}_{22}^i \bar{g}_{44}^i - (\bar{g}_{24}^i)^2} , \quad (44)$$

$$B = \frac{1}{\bar{h}} \sum h_i \frac{H_3^i g_{24} - H_2^i \bar{g}_{44}^i}{\bar{g}_{22}^i \bar{g}_{44}^i - (\bar{g}_{24}^i)^2} \quad (45)$$

$$C = \frac{1}{\bar{h}} \sum \frac{h_i}{g_{33}^i} \left[\frac{g_{23} (H_3^i g_{24} - H_2^i g_{44})}{\bar{g}_{22}^i \bar{g}_{44}^i - (\bar{g}_{24}^i)^2} - \frac{g_{34} (H_2^i g_{24} - H_3^i g_{22})}{(\bar{g}_{24}^i)^2} \right] \quad (46)$$

$$D = \frac{1}{\bar{h}} \sum h_i \frac{g_{24}^i}{\bar{g}_{22}^i \bar{g}_{44}^i - (\bar{g}_{24}^i)^2} \quad (47)$$

$$\frac{1}{\bar{h}} \sum \frac{h_i}{g_{33}^i} \frac{g_{23}^i g_{24}^i - g_{34}^i \bar{g}_{22}^i}{\bar{g}_{22}^i \bar{g}_{44}^i - (\bar{g}_{24}^i)^2}$$

$$\frac{1}{\bar{h}} \sum h_i \frac{g_{22}^i}{\bar{g}_{22}^i \bar{g}_{44}^i - (\bar{g}_{24}^i)^2}$$

$$\frac{1}{\bar{h}} \sum h_i \frac{g_{24}^i \bar{\alpha}_4^i - g_{44}^i \bar{\alpha}_2^i}{\bar{g}_{22}^i \bar{g}_{44}^i - (\bar{g}_{24}^i)^2} \quad (50)$$

$$Q = \frac{1}{\bar{h}} \sum \frac{h_i}{g_{33}^i} \left[\frac{g_{23} (g_{24}^i \bar{\alpha}_4^i - g_{44}^i \bar{\alpha}_2^i)}{\bar{g}_{22}^i \bar{g}_{44}^i - (\bar{g}_{24}^i)^2} - \frac{g_{34} (g_{24}^i \bar{\alpha}_2^i - g_{22}^i \bar{\alpha}_4^i)}{(\bar{g}_{24}^i)^2} \right] \alpha_3^i \quad (51)$$

$$R = \frac{1}{\bar{h}} \sum h_i \frac{g_{24}^i \bar{\alpha}_2^i - g_{22}^i \bar{\alpha}_4^i}{\bar{g}_{22}^i \bar{g}_{44}^i - (\bar{g}_{24}^i)^2} \quad (52)$$

$$U = \sum h_i \frac{g_{24}^i \bar{\beta}_4^i - g_{44}^i \bar{\beta}_2^i}{\bar{g}_{22}^i \bar{g}_{44}^i - (\bar{g}_{24}^i)^2} \quad (53)$$

$$V = \sum \frac{h_i}{g_{33}^i} \left[\frac{g_{23}^i (\bar{g}_{24}^i \bar{\beta}_4^i - \bar{g}_{44}^i \bar{\beta}_2^i) + g_{34}^i (\bar{g}_{24}^i \bar{\beta}_2^i - \bar{g}_{22}^i \bar{\beta}_4^i)}{\bar{g}_{22}^i \bar{g}_{44}^i - (\bar{g}_{24}^i)^2} \right] \bar{\beta}_3^i$$

$$W = \frac{1}{h} \sum_{i=1}^{N+1} h_i \frac{\bar{g}_{24}^i \bar{\beta}_2^i - \bar{g}_{22}^i \bar{\beta}_4^i}{\bar{g}_{22}^i \bar{g}_{44}^i - (\bar{g}_{24}^i)^2}, \quad (55)$$

$$X = - \frac{1}{h} \sum_{i=1}^{N+1} h_i \frac{\bar{g}_{24}^i}{\bar{g}_{22}^i \bar{g}_{44}^i - (\bar{g}_{24}^i)^2} = D, \quad (56)$$

$$Y = \frac{1}{h} \sum_{i=1}^{N+1} h_i \frac{H_2^i \bar{g}_{24}^i - H_3^i \bar{g}_{22}^i}{\bar{g}_{22}^i \bar{g}_{44}^i - (\bar{g}_{24}^i)^2} = E. \quad (57)$$

The relations (56) and (57) indicate that the stress/strain relations (41-43) exhibit the required symmetry of elastic constants. The relations (41-43) relating the effective applied stresses σ , σ^* and τ to the uniform strains ϵ , ϵ^* and γ enable the hygro-thermo-elastic constants of an uncracked laminate to be determined. In Appendix B the inverse form of the relations (41-43) is derived showing how familiar hygro-thermo-elastic constants for the laminate may be related to the various parameters defined by (44-57).

5. Equilibrium equations

For a cracked laminate subject to generalised plane strain conditions it follows, from the fact that the stress field is independent of z , that the equilibrium equations (1-3) reduce to the following form

$$\frac{\partial \sigma_{xx}^i}{\partial x} + \frac{\partial \sigma_{xy}^i}{\partial y} = 0, \quad (58)$$

$$\frac{\partial \sigma_{xy}^i}{\partial x} + \frac{\partial \sigma_{yy}^i}{\partial y} = 0, \quad (59)$$

$$\frac{\partial \sigma_{xz}^i}{\partial x} + \frac{\partial \sigma_{yz}^i}{\partial y} = 0. \quad (60)$$

6. Representation of stress and displacement fields

The fundamental assumption of the analysis for cracked laminates is that the shear stress components σ_{xy} and σ_{xz} have the following piece-wise linear forms (in x), for $i = 1 \dots N+1$

$$\sigma_{xy}^i = p_i'(y) \frac{x - x_{i-1}}{h_i} - p_{i-1}'(y) \frac{x - x_i}{h_i}, \quad x_{i-1} \leq x \leq x_i, \quad (61)$$

$$\sigma_{xz}^i = q_i'(y) \frac{x - x_{i-1}}{h_i} - q_{i-1}'(y) \frac{x - x_i}{h_i}, \quad x_{i-1} \leq x \leq x_i, \quad (62)$$

where $p_i(y)$, and $q_i(y)$, $i = 1 \dots N+1$, are functions only of y that are to be determined, and which have the property that they are identically zero for the case of undamaged laminates. The derivatives of these functions are denoted by primes. The form of the relations (61) and (62) ensures that the stress components σ_{xy} and σ_{xz} are automatically continuous across the interfaces. From (61) and (62) it is clear that $p_i'(y)$ and $q_i'(y)$ define the distributions of the stress components σ_{xy}^i and σ_{xz}^i on the interface at $x = x_i$. It follows from (58) and (61) that, for $i = 1 \dots N+1$

$$\sigma_{xx}^i = \frac{x - x_i}{2h_i} \left[(x - x_i) p_{i-1}''(y) - (x - x_i + 2h_i) p_i''(y) \right] + S_i(y), \quad x_{i-1} \leq x \leq x_i, \quad (63)$$

where the function $S_i(y)$ arises from the integration with respect to x , and will be defined later.

It follows from (59-62) that the stress components σ_{yy} and σ_{yz} are of the following form independent of the variable x , for $x_{i-1} < x \leq x_i$ and $i = 1 \dots N+1$

$$\sigma_{yy}^i = \frac{p_{i-1}(y) - p_i(y)}{h_i} + \sigma_i, \quad (64)$$

$$\sigma_{yz}^i = \frac{q_{i-1}(y) - q_i(y)}{h_i} + \tau_i, \quad (65)$$

where σ_i and τ_i are the uniform axial and shear stresses that would result if the laminate were undamaged and subject to the same loading conditions. The representation for the stress field defined by (61-65) satisfies the equilibrium equations (58-60) for any functions $p_i(y)$ and $q_i(y)$, $i = 1 \dots N+1$, characterising stress transfer in the cracked laminate.

It follows from (29) and (63-65) that, for $x_{i-1} < x \leq x_i$ and $i = 1 \dots N+1$,

$$u_i = \frac{(x - x_i)^2}{6h_i} \bar{g}_{11}^i \left[(x - x_i) p_{i-1}''(y) - (x - x_i + 3h_i) p_i''(y) \right] + (x - x_i) \left[\bar{g}_{11}^i S_i(y) + \bar{g}_{12}^i \frac{p_{i-1}(y) - p_i(y)}{h_i} + \bar{g}_{14}^i \frac{q_{i-1}(y) - q_i(y)}{h_i} + \bar{\epsilon}_i \right] + U_i(y), \quad (66)$$

where $u_i(x, y) = U_i(y)$, where $U_i(y)$, arising from the integration with respect to x , is the through-thickness displacement (to be defined later) at the interface $x = x_i$, and where $\bar{\epsilon}_i$ is

the through-thickness strain in the i^{th} layer of an undamaged laminate (subject to longitudinal, transverse and shear applied stresses σ , σ' and τ respectively) defined by

$$\xi_i = \bar{g}_{12}^i \sigma_i + \bar{g}_{14}^i \tau_i + \bar{\alpha}_1^i \Delta T + \bar{\beta}_1^i \Delta c + H_1^i \epsilon_c^* , \quad i = 1 \dots N+1 , \quad (67)$$

which is identical to (35) when ϵ_c^* is replaced by ϵ^* . It follows from (8), (61), (62) and (66) that the expression for v_i , for $x_{i-1} < x \leq x_i$ and $i = 1 \dots N+1$, is given by

$$\begin{aligned} v_i = & \frac{x - x_i}{h_i} (x - x_i + 2h_i) \left[a_{11}^i p_i'(y) + a_{12}^i q_i'(y) \right] - \frac{(x - x_i)^2}{h_i} \left[a_{11}^i p_{i-1}'(y) + a_{12}^i q_{i-1}'(y) \right] \\ & - \frac{(x - x_i)^3}{24h_i} \bar{g}_{11}^i \left[(x - x_i) p_{i-1}'''(y) - (x - x_i + 4h_i) p_i'''(y) \right] \\ & - \frac{1}{2} (x - x_i)^2 \left[\bar{g}_{11}^i S_i'(y) + \bar{g}_{12}^i \frac{p_{i-1}'(y) - p_i'(y)}{h_i} + \bar{g}_{14}^i \frac{q_{i-1}'(y) - q_i'(y)}{h_i} \right] \\ & - (x - x_i) U_i'(y) + V_i(y) , \end{aligned} \quad (68)$$

where $v_i(x_i, y) = V_i(y)$, where $V_i(y)$ (to be defined later), arising from the integration with respect to x , is the longitudinal displacement distribution at $x = x_i$, $i = 1 \dots N+1$.

Similarly from (9), (61), (62) and (66), the expression for w_i , for $x_{i-1} < x \leq x_i$ and $i = 1 \dots N+1$, is given by

$$\begin{aligned} w_i = & \frac{x - x_i}{h_i} (x - x_i + 2h_i) \left[a_{12}^i p_i'(y) + a_{22}^i q_i'(y) \right] \\ & - \frac{(x - x_i)^2}{h_i} \left[a_{12}^i p_{i-1}'(y) + a_{22}^i q_{i-1}'(y) \right] + W_i(y) + \epsilon_c^* z \end{aligned} \quad (69)$$

where $w_i(x_i, y) = W_i(y) + \epsilon_c^* z$ is consistent with the form of (27)₃, and where $W_i(y)$ is to be defined later.

7. Boundary and interface conditions

Fig.1 shows the region under consideration in the x - y plane for stress transfer between the $N+1$ layers. Only solutions symmetric about $x = 0$ and $y = 0$ are considered in the region $0 \leq x \leq h$, $0 \leq y \leq L$. Symmetry about $x = 0$ for all y asserts that

$$\sigma_{xy}^1 = 0 , \quad \sigma_{xz}^1 = 0 , \quad u_1 = 0 , \quad \text{on } x = 0 . \quad (70)$$

For the assumed perfect bonding on the N interfaces, it is essential that for all y

$$\sigma_{xx}^i = \sigma_{xx}^{i+1} , \quad \sigma_{xy}^i = \sigma_{xy}^{i+1} , \quad \sigma_{xz}^i = \sigma_{xz}^{i+1} \quad \text{on } x = x_i , \quad i = 1 \dots N , \quad (71)$$

and

$$u_i = u_{i+1}, \quad v_i = v_{i+1}, \quad w_i = w_{i+1}, \quad \text{on } x = x_i, \quad i = 1 \dots N. \quad (72)$$

The external surface of the outer layer is stress-free, so that for all y

$$\sigma_{xx}^{N+1} = 0, \quad \sigma_{xy}^{N+1} = 0, \quad \sigma_{xz}^{N+1} = 0 \quad \text{on } x = x_{N+1} = h. \quad (73)$$

Any undamaged layer surfaces on $y = \pm L$ are subject to the boundary conditions

$$\begin{aligned} v_i(x, \pm L) &= \pm \epsilon_c L, \quad w_i(x, \pm L, z) = \pm \gamma_c L + \epsilon_c^* z, \quad \text{for all } x, z, \\ \pm L) &= 0, \quad \text{for all } x, \end{aligned} \quad (74)$$

where ϵ_c is the effective axial strain of the cracked laminate, where ϵ_c^* is the uniform transverse strain, and where γ_c is the effective in-plane shear strain applied to the laminate. Any cracked surfaces on $y = \pm L$ will be stress free so that

$$\sigma_{yy}^i(x, \pm L) = 0, \quad \sigma_{xy}^i(x, \pm L) = 0, \quad \sigma_{yz}^i(x, \pm L) = 0, \quad \text{for all } x. \quad (75)$$

8. Recurrence relations

The functions S_i , U_i , V_i and W_i appearing respectively in the expressions (63), (66), (68) and (69) are the values of σ_{xx}^i , u_i , v_i and w_i (with $z = 0$) respectively at the interfaces $x = x_i$, for $i = 1 \dots N$ and at the external surface $x = x_{N+1}$. These definitions can be extended without loss of generality so that S_0 , U_0 , V_0 and W_0 are the values of these variables at $x = x_0 = 0$.

It follows on substituting $x = x_{i-1}$ in (63) that

$$S_{i-1}(y) = \frac{h_i}{2} [p_{i-1}''(y) + p_i''(y)] + S_i(y), \quad \text{for } i = N+1 \dots 1, \quad \text{where } S_{N+1}(y) \equiv 0, \quad (76)$$

since from (73)₁ $\sigma_{xx} = 0$ on $x = h$. On substituting $x = x_{i-1}$ in (66) the following recurrence relation is derived for the functions $U_i(y)$

$$\begin{aligned} U_i(y) &= \frac{h_i^2}{6} \bar{g}_{11}^i [p_{i-1}''(y) + 2p_i''(y)] \\ &+ h_i \left[\bar{g}_{11}^i S_i(y) + \bar{g}_{12}^i \frac{p_{i-1}(y) - p_i(y)}{h_i} + \bar{g}_{14}^i \frac{q_{i-1}(y) - q_i(y)}{h_i} + \xi_i \right] + U_{i-1}(y), \quad (77) \\ &\text{for } i = 1 \dots N+1, \quad \text{where } U_0(y) \equiv 0, \end{aligned}$$

since from (70)₃ $u = 0$ on $x = 0$. On substituting $x = x_{i-1}$ in (68) the following recurrence relation can be derived for the functions $\Delta V_i(y) \equiv V_i(y) - V_{N+1}(y)$

$$\begin{aligned}
\Delta V_{i-1}(y) = & -h_i \left[a_{11}^i p_i'(y) + a_{12}^i q_i'(y) \right] - h_i \left[a_{11}^i p_{i-1}'(y) + a_{12}^i q_{i-1}'(y) \right] \\
& - \frac{h_i^3}{24} \bar{g}_{11}^i \left[p_{i-1}'''(y) + 3 p_i'''(y) \right] \\
& - \frac{h_i^2}{2} \left[\bar{g}_{11}^i S_i'(y) + \bar{g}_{12}^i \frac{p_{i-1}'(y) - p_i'(y)}{h_i} + \bar{g}_{14}^i \frac{q_{i-1}'(y) - q_i'(y)}{h_i} \right] \\
& + h_i U_i'(y) + \Delta V_i(y), \quad \text{for } i = N+1 \dots 2, \quad \text{where } \Delta V_{N+1}(y) \equiv 0
\end{aligned} \tag{78}$$

On substituting $x = x_{i-1}$ in (69) the following recurrence relation for the functions $\Delta W_i(y) \equiv W_i(y) - W_{N+1}(y)$ can be derived

$$\begin{aligned}
\Delta W_{i-1}(y) = & -h_i \left[a_{12}^i p_i'(y) + a_{22}^i q_i'(y) \right] - h_i \left[a_{12}^i p_{i-1}'(y) + a_{22}^i q_{i-1}'(y) \right] + \Delta W_i(y), \\
& \text{for } i = N+1 \dots 2, \quad \text{where } \Delta W_{N+1}(y) \equiv 0.
\end{aligned} \tag{79}$$

Because symmetry is assumed about $x = 0$, and since $p_0(y)$ and $q_0(y)$ must have zero values for uncracked laminates, the relations (61) and (62), and the boundary conditions (70)₁ and (70)₂ assert that the functions p_0 and q_0 must be such that

$$p_0(y) \equiv 0, \quad q_0(y) \equiv 0. \tag{80}$$

Similarly the relations (61) and (62), and the boundary conditions (73)₂ and (73)₃ assert that

$$p_{N+1}(y) \equiv 0, \quad q_{N+1}(y) \equiv 0. \tag{81}$$

The expressions derived for the stress and displacement components satisfy exactly the equilibrium equations (58-60) together with the stress-strain relations (4), (6), (8) and (9) for *any* functions $p_i(y)$ and $q_i(y)$, $i = 1 \dots N$. In addition, provided that the functions $S_i(y)$, $U_i(y)$, $V_i(y)$ and $W_i(y)$ are generated respectively by the recurrence relations (76-79), the stress and displacement fields automatically satisfy the interface conditions (71) and (72) and some of the other boundary conditions. The remaining stress-strain relations (5) and (7) (and (30) and (31)) can only be satisfied in an average sense that will be described in the next section.

9. Governing differential equations

When deriving the displacement distributions given by (66), (68) and (69), use has been made of the stress-strain relations (8), (9) and (29) in such a way that these relations are automatically satisfied by the displacement field. It is necessary, therefore, to consider the remaining stress-strain relations given by (30) and (31) for the in-plane strains ϵ_{yy} and ϵ_{yz} . As it is not possible to satisfy these relations exactly using the stress and displacement representation, an attempt is made to satisfy them approximately by averaging these stress-strain relations through the thickness of each layer. By making use of ply refinement techniques the thickness of each layer can be made small enough for there to be minimal error when replacing (30) and (31) by their through-thickness averages. The average of any quantity $f_i(x, y, z)$ through the thickness of the i^{th} layer is denoted by an overbar and defined by

$$\bar{f}_i(y, z) = \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f_i(x, y, z) dx, \quad h_i = x_i - x_{i-1}. \quad (82)$$

The through-thickness averages of the stress-strain relations (30) and (31) are written for $i = 1 \dots N+1$

$$\bar{\epsilon}_{yy}^i \equiv \frac{\partial \bar{v}_i}{\partial y} = \bar{g}_{12}^i \bar{\sigma}_{xx}^i + \bar{g}_{22}^i \bar{\sigma}_{yy}^i + \bar{g}_{24}^i \bar{\sigma}_{yz}^i + \bar{\alpha}_2^i \Delta T + \bar{\beta}_2^i \Delta c + \frac{\bar{g}_{23}^i}{\bar{g}_{33}^i} \epsilon_c^*, \quad (83)$$

$$2\bar{\epsilon}_{yz}^i \equiv \frac{\partial \bar{v}_i}{\partial z} + \frac{\partial \bar{w}_i}{\partial y} = \bar{g}_{14}^i \bar{\sigma}_{xx}^i + \bar{g}_{24}^i \bar{\sigma}_{yy}^i + \bar{g}_{44}^i \bar{\sigma}_{yz}^i + \bar{\alpha}_4^i \Delta T + \bar{\beta}_4^i \Delta c + \frac{\bar{g}_{34}^i}{\bar{g}_{33}^i} \epsilon_c^*. \quad (84)$$

From (63-65) it is clear that on averaging

$$\bar{\sigma}_{xx}^i = \frac{h_i}{6} [p_{i-1}''(y) + 2p_i''(y)] + S_i(y), \quad (85)$$

$$\bar{\sigma}_{yy}^i \equiv \sigma_{yy}^i = \frac{p_{i-1}(y) - p_i(y)}{h_i} + \sigma_i, \quad \bar{\sigma}_{yz}^i \equiv \sigma_{yz}^i = \frac{q_{i-1}(y) - q_i(y)}{h_i} + \tau_i. \quad (86)$$

Now from (68) the average of the displacement component v_i is given by

$$\begin{aligned} \bar{v}_i(y) = & -\frac{2h_i}{3} [a_{11}^i p_i'(y) + a_{12}^i q_i'(y)] - \frac{h_i}{3} [a_{11}^i p_{i-1}'(y) + a_{12}^i q_{i-1}'(y)] \\ & + \frac{h_i^3}{120} \bar{g}_{11}^i [p_{i-1}'''(y) + 4p_i'''(y)] \\ & - \frac{h_i^2}{6} \left[\bar{g}_{11}^i S_i'(y) + \bar{g}_{12}^i \frac{p_{i-1}'(y) - p_i'(y)}{h_i} + \bar{g}_{14}^i \frac{q_{i-1}'(y) - q_i'(y)}{h_i} \right] \\ & + \frac{h_i}{2} U_i'(y) + \Delta V_i(y) + V_{N+1}(y). \end{aligned} \quad (87)$$

From (69) the average of the displacement component w_i is given by

$$\begin{aligned} \bar{w}_i(y, z) = & -\frac{2h_i}{3} [a_{12}^i p_i'(y) + a_{22}^i q_i'(y)] - \frac{h_i}{3} [a_{12}^i p_{i-1}'(y) + a_{22}^i q_{i-1}'(y)] \\ & + \Delta W_i(y) + W_{N+1}(y) + \epsilon_c^* z. \end{aligned} \quad (88)$$

The required governing differential equations are obtained by substituting (87) and (88) into (83) and (84) using (85) and (86) having determined the functions $S_i(y)$, $U_i(y)$, $\Delta V_i(y)$ and $\Delta W_i(y)$ using (76-79), and then eliminating the functions $V_{N+1}(y)$ and $W_{N+1}(y)$ by subtraction. It can be shown that the resulting differential equations are of the following form

$$\begin{aligned}
& \sum_{i=1}^N F_{ij} p_i^{(4)}(y) + \sum_{i=1}^N G_{ij} p_i^{(2)}(y) + \sum_{i=1}^N H_{ij} p_i(y) \\
& + \sum_{i=1}^N g_{ij} q_i^{(2)}(y) + \sum_{i=1}^N h_{ij} q_i(y) = 0, \quad j = 1 \dots N,
\end{aligned} \tag{89}$$

$$\begin{aligned}
& \sum_{i=1}^N A_{ij} p_i^{(2)}(y) + \sum_{i=1}^N B_{ij} p_i(y) \\
& + \sum_{i=1}^N a_{ij} q_i^{(2)}(y) + \sum_{i=1}^N b_{ij} q_i(y) = 0, \quad j = 1 \dots N
\end{aligned} \tag{90}$$

where the coefficients in (89) and (90) are best determined numerically using the technique described in [12] for the case of cross-ply laminates. In (89) and (90) $p^{(n)}(y)$ and $q^{(n)}(y)$, $n \geq 1$, denote the n^{th} derivatives of the functions $p(y)$ and $q(y)$ respectively. The method of solving numerically the system of differential equations (89) and (90) is described in [18]. Once the solution of these differential equations is known satisfying appropriate boundary conditions, the functions $V_{N+1}(y)$ and $W_{N+1}(y)$ are calculated using (83), (84), (87) and (88). The functions $V_i(y)$, $W_i(y)$, $i = 1 \dots N$ can then be found having used the recurrence relations (78) and (79) to determine values of ΔV_i , ΔW_i , $i = 1 \dots N$. It should be noted that when the model is applied to a cross-ply laminate the coefficients g_{ij} , h_{ij} , A_{ij} and B_{ij} are all zero with the result that the simultaneous set of differential equations given by (89) and (90) separates into two independent systems of equations. The numerical solution technique [18] used to solve the system (89) and (90) cannot be used to solve the equations that result for the case of cross-ply laminates. Cross-ply laminates are solved using an alternative technique similar to that used for axi-symmetric stress transfer problems [17].

10. Average in-plane displacements

The expressions (87) and (88) for the through-thickness averages of the in-plane displacements v_i and w_i can be simplified as follows. On substituting (87) in (83) the resulting equation can be solved to provide an expression for the quantity $V_i'(y)$. On integrating (83) with respect to y the following simplified expression is obtained

$$\begin{aligned}
\bar{v}_i(y) = & \frac{h_i}{6} \bar{g}_{12}^i \left[p_{i-1}'(y) + 2p_i'(y) \right] + \frac{\bar{g}_{22}^i}{h_i} \left[p_{i-1}^*(y) - p_i^*(y) \right] \\
& + \frac{\bar{g}_{24}^i}{h_i} \left[q_{i-1}^*(y) - q_i^*(y) \right] + \bar{g}_{12}^i S_i^*(y) + \varepsilon y,
\end{aligned} \tag{91}$$

where use has been made of (36) on setting $\varepsilon^* = \varepsilon_c^*$, and where

$$\begin{aligned}
 p_i^*(y) &= \int_0^y p_i(s) ds, & q_i^*(y) &= \int_0^y q_i(s) ds, \\
 S_i^*(y) &= \int_0^y S_i(s) ds,
 \end{aligned} \tag{92}$$

$$\varepsilon = \bar{g}_{22}^i \sigma_i + \bar{g}_{24}^i \tau_i + \bar{\alpha}_2^i \Delta T + \bar{\beta}_2^i \Delta c + H_2^i \varepsilon_c^*.$$

On substituting (88) in (84) the resulting equation can be solved to provide an expression for the quantity $W_i'(y)$. On integrating (84) with respect to y the following simplified expression can be derived

$$\begin{aligned}
 \bar{w}_i(y, z) &= \frac{h_i}{6} \bar{g}_{14}^i [p_{i-1}'(y) + 2p_i'(y)] + \frac{\bar{g}_{24}^i}{h_i} [p_{i-1}^*(y) - p_i^*(y)] \\
 &+ \frac{\bar{g}_{44}^i}{h_i} [q_{i-1}^*(y) - q_i^*(y)] + \bar{g}_{14}^i S_i^*(y) + \gamma y + \varepsilon_c^* z,
 \end{aligned} \tag{93}$$

$$\text{where} \quad \gamma = \bar{g}_{24}^i \sigma_i + \bar{g}_{44}^i \tau_i + \bar{\alpha}_4^i \Delta T + \bar{\beta}_4^i \Delta c + H_3^i \varepsilon_c^*,$$

where use has been made of (37) on setting $\varepsilon^* = \varepsilon_c^*$.

11. Effective stresses for cracked laminates

Consider a cracked laminate having width $2W$. On multiplying (64) and (65) by h_i and summing over i it follows that the effective axial and shear stresses on $y = \pm L$ are given by

$$\begin{aligned}
 \sigma &= \frac{1}{2hW} \sum_{i=1}^{N+1} \int_{x_{i-1}}^{x_i} \int_{-W}^W \sigma_{yy}^i dz dx = \frac{1}{h} \sum_{i=1}^{N+1} h_i \sigma_i, \\
 \tau &= \frac{1}{2hW} \sum_{i=1}^{N+1} \int_{x_{i-1}}^{x_i} \int_{-W}^W \sigma_{yz}^i dz dx = \frac{1}{h} \sum_{i=1}^{N+1} h_i \tau_i,
 \end{aligned} \tag{94}$$

where use has been made of the fact that $p_0(y) \equiv 0$, $p_{N+1}(y) \equiv 0$, and of the relations (41) and (43). It is clear from (64) and (65) that the effective axial and shear stresses for a cracked laminate are identical to the effective axial and shear stresses for the corresponding uncracked laminate. The effective transverse stress applied to the cracked laminate is defined by

$$\sigma_i = \frac{1}{2hL} \sum_{i=1}^{N+1} \int_{x_{i-1}}^{x_i} \int_{-L}^L \sigma_{zz}^i dy dx . \quad (95)$$

On using (82), the definition (95) may be written

$$\sigma_i = \frac{1}{2L} \sum_{i=1}^{N+1} \frac{h_i}{h} \int_{-L}^L \bar{\sigma}_{zz}^i dy , \quad (96)$$

where $\bar{\sigma}_{zz}^i$ is obtained from the through-thickness average of (28). Since $\sigma_{xy} = 0$ on $y = \pm L$, it follows from (61), (80) and (81) that $p_i'(\pm L) = 0$, $i = 0 \dots N+1$. Now from (76) and (85) it can then be shown that

$$\frac{1}{2L} \int_{-L}^L \bar{\sigma}_{xx}^i(y) dy = 0 . \quad (97)$$

From (64), (65) and (92) it can be shown that

$$\frac{1}{2L} \int_{-L}^L \bar{\sigma}_{yy}^i(y) dy = \frac{1}{h_i L} [p_{i-1}^*(L) - p_i^*(L)] + \sigma_i , \quad (98)$$

$$\frac{1}{2L} \int_{-L}^L \bar{\sigma}_{yz}^i(y) dy = \frac{1}{h_i L} [q_{i-1}^*(L) - q_i^*(L)] + \tau_i . \quad (99)$$

On substituting (97-99) into the average of (28) followed by substitution in (96) it can be shown that

$$\begin{aligned} \sigma_i = \sum_{i=1}^{N+1} \frac{h_i}{h} \cdot \frac{1}{g_{33}^i} & \left[\epsilon_c^* - \alpha_3^i \Delta T - \beta_3^i \Delta c - g_{23}^i \left(\frac{1}{h_i L} (p_{i-1}^*(L) - p_i^*(L)) + \sigma_i \right) \right. \\ & \left. - g_{34}^i \left(\frac{1}{h_i L} (q_{i-1}^*(L) - q_i^*(L)) + \tau_i \right) \right] \end{aligned} \quad (100)$$

Thus the relation (100) indicates how the effective transverse stress applied to the cracked laminate may be calculated in terms of the uniform transverse strain ϵ_c^* . It should be noted that the effective shear stress on $z = \pm W$ is τ , consistent with (94).

12. The role of applied strains for cracked laminates

The role of the applied strains for a cracked laminate is best illustrated by considering the following quantity that arises when estimating the potential energy of the applied tractions, namely

$$\int_{\epsilon} n_i \sigma_{ij} u_j dS , \quad (101)$$

where σ_{ij} and u_j are the stress and displacement fields for that region of the cracked laminate that is enclosed by the surface S having a unit outward normal n_i . As the faces of the laminate are assumed to be stress free the region S can be taken as the surfaces defining the edges of the laminate. Considering the edge on $y = L$ for $-h < x < h$, the potential energy of the applied tractions is given by

$$2 \sum_{i=1}^{N+1} \int_{x_{i-1}}^{x_i} \int_{-W}^W \sigma_{yy}^i(x, L, z) v_i(x, L, z) dz dx + 2 \sum_{i=1}^{N+1} \int_{x_{i-1}}^{x_i} \int_{-W}^W \sigma_{yz}^i(x, L, z) w_i(x, L, z) dz dx . \quad (102)$$

From (64) and (65) the stress components σ_{yy}^i and σ_{yz}^i are independent of x and z , and from (68) v_i is independent of z , so that (102) may be written on using (82) as

$$2 \sum_{i=1}^{N+1} h_i \int_{-W}^W \sigma_{yy}^i(L) \bar{v}_i(L) dz + 2 \sum_{i=1}^{N+1} h_i \int_{-W}^W \sigma_{yz}^i(L) \bar{w}_i(L, z) dz . \quad (103)$$

Now for those layers that are uncracked on $y = L$ it follows from (74) that

$$\bar{v}_i(L) = \epsilon_c L , \quad \bar{w}_i(L, z) = \gamma_c L + \epsilon_c^* z . \quad (104)$$

Thus on substituting in (103) the potential energy of the tractions applied on $y = L$ may be written

$$2 L \epsilon_c \sum_{i=1}^{N+1} h_i \int_{-W}^W \sigma_{yy}^i(L) dz + 2 L \gamma_c \sum_{i=1}^{N+1} h_i \int_{-W}^W \sigma_{yz}^i(L) dz . \quad (105)$$

On using (94) this reduces to the simple result

$$4hLW [\sigma \epsilon_c + \tau \gamma_c] . \quad (106)$$

Considering the edge on $z = W$ for $-h < x < h$, the potential energy of the applied tractions is given by

$$2 \sum_{i=1}^{N+1} \int_{x_{i-1}}^{x_i} \int_{-L}^L \sigma_{yz}^i(x, y, W) v_i(x, y, W) dy dx + 2 \sum_{i=1}^{N+1} \int_{x_{i-1}}^{x_i} \int_{-L}^L \sigma_{zz}^i(x, y, W) w_i(x, y, W) dy dx \quad (107)$$

Now if $f(y) = f(-y)$ so that f is any even function of y , and $g(y) = -g(-y)$ so that g is any odd function of y , then

$$\int_{-L}^L f(y) g(y) dy = 0 . \quad (108)$$

Since σ_{yz}^i is an even function of y and v_i is an odd function of y , it follows immediately that the first term on the R.H.S. of (107) is zero. Because $w_i - \epsilon_c^* z$ is an odd function of y , and because σ_{xx}^i , σ_{yy}^i , σ_{zz}^i and hence from (28) σ_{zz}^i are even functions of y , it follows using (95) and (108) that the second term on the R.H.S. of (107) has the value

$$4hLW \sigma_i \epsilon_c^* . \quad (109)$$

Thus on adding (106) and (109) and combining with an equal contribution from the surfaces $y = -L$ and $z = -W$, the potential energy of the applied tractions per unit volume may be written in the compact form

$$\frac{1}{V} \int_S n_i \sigma_{ij} u_j dS = \sigma \epsilon_c + \sigma_i \epsilon_c^* + \tau \gamma_c , \quad (110)$$

where $V = 8hLW$ is the volume of the region of laminate being considered. The analysis presented above is valid for general symmetric multiple ply laminates subject to ply cracking in one orientation such that the fibres of cracked plies lie in the z -direction. The analysis clearly illustrates the important role of the effective applied strains for a cracked laminate that appear in the boundary conditions (74).

13. Boundary conditions for the differential equations

The objective now is to specify boundary conditions that will lead to a unique solution of the system of differential equations specified by (89) and (90). Such a solution can then be used to calculate the stress and displacement fields everywhere in the laminate using the relations (61-69). Attention is restricted to the case where there is symmetry about the plane $y = 0$ so that only 3N boundary conditions are required. For this case the functions $p_i(y)$ and $q_i(y)$ are symmetric about $y = 0$ and it is necessary to impose the boundary conditions only at $y = L$.

The loading is assumed to be such that the uniform in-plane strains in an undamaged laminate are ϵ , ϵ^* and γ . The parameters ΔT and Δc describing the hygro-thermal conditions are assumed to be specified. The corresponding effective stresses σ , σ^* and τ are calculated directly using (38-43). When a cracked laminate is subject to the same applied stresses the effective in-plane strains are ϵ_v , ϵ_c^* and γ_c . The values of these parameters are unknown and must be found as part of the solution of the stress transfer problem.

From (74) and (75), the shear stress σ_{xy} is zero everywhere on the plane $y = L$. Thus from (61), (80) and (81)

$$p_i'(L) = 0 , \quad i = 1 \dots N . \quad (111)$$

On $y = L$ it follows from (75) that for cracked layers the normal stress σ_{yy} and shear stress σ_{yz} are zero so that from (64) and (65) the following boundary conditions are obtained

$$p_i(L) - p_{i-1}(L) = h_i \sigma_i , \quad (112)$$

$$q_i(L) - q_{i-1}(L) = h_i \tau_i , \quad (113)$$

where σ_i and τ_i are given by (39) and (40) and where i (taking values appropriate for cracked layers) lies in the range $1 \leq i \leq N+1$.

Consider the boundary conditions that must be applied to **uncracked layers** in the laminate. Use is made of the boundary conditions (74) relating to displacement components for uncracked layers. As these conditions cannot be satisfied exactly by the relations (68) and (69) because of the x dependence, the following less restrictive equivalent average conditions for uncracked layers are imposed on $y = L$, namely

$$\bar{v}_i(L) = L\epsilon_c, \quad \bar{w}_i(L, z) = L\gamma_c + \epsilon_c^* z, \quad (114)$$

where $\bar{v}_i$ and $\bar{w}_i$ are defined by (91) and (93), and where i (taking values appropriate for uncracked layers) lies in range $1 \leq i \leq N+1$. When use is made of (76), (111), (91) and (93) together with the symmetry of the functions $p_i(y)$ and $q_i(y)$ about $y = 0$, the conditions (114) for **uncracked layers** may be expressed

$$\epsilon_c = \frac{\bar{g}_{22}^i}{h_i L} [p_{i-1}^*(L) - p_i^*(L)] + \frac{\bar{g}_{24}^i}{h_i L} [q_{i-1}^*(L) - q_i^*(L)] + \epsilon, \quad (115)$$

$$\gamma_c = \frac{\bar{g}_{24}^i}{h_i L} [p_{i-1}^*(L) - p_i^*(L)] + \frac{\bar{g}_{44}^i}{h_i L} [q_{i-1}^*(L) - q_i^*(L)] + \gamma, \quad (116)$$

where i (taking values appropriate for uncracked layers) lies in the range $1 \leq i \leq N+1$. The parameters ϵ_c and γ_c are not yet known and therefore must be eliminated from these boundary conditions. It is convenient to let j be the label of the nearest uncracked layer to the centre plane of the laminate. It then follows by subtraction that the required boundary conditions for **uncracked layers** derived from (115) and (116) are of the form

$$\begin{aligned} & \frac{\bar{g}_{22}^i}{h_i L} [p_{i-1}^*(L) - p_i^*(L)] - \frac{\bar{g}_{22}^j}{h_j L} [p_{j-1}^*(L) - p_j^*(L)] \\ & + \frac{\bar{g}_{24}^i}{h_i L} [q_{i-1}^*(L) - q_i^*(L)] - \frac{\bar{g}_{24}^j}{h_j L} [q_{j-1}^*(L) - q_j^*(L)] = 0, \\ & \frac{\bar{g}_{24}^i}{h_i L} [p_{i-1}^*(L) - p_i^*(L)] - \frac{\bar{g}_{24}^j}{h_j L} [p_{j-1}^*(L) - p_j^*(L)] \\ & + \frac{\bar{g}_{44}^i}{h_i L} [q_{i-1}^*(L) - q_i^*(L)] - \frac{\bar{g}_{44}^j}{h_j L} [q_{j-1}^*(L) - q_j^*(L)] = 0, \end{aligned} \quad (118)$$

where i and j (taking values appropriate for uncracked layers such that $i \neq j$) lie in the range $1 \leq i, j \leq N+1$.

The conditions (111-113), (117) and (118) provide the $3N$ boundary conditions required to ensure that the system of differential equations given by (89) and (90) has a unique solution that is symmetric about $y = 0$.

14. Calculation of elastic constants

The method of estimating the elastic constants assumes initially that the parameters σ , τ , ϵ_c^* , ΔT and Δc are specified. The axial strain ϵ , the shear strain γ and the effective transverse stress σ' for an uncracked laminate, the corresponding stresses σ_i , τ_i and the through-thickness strains ϵ_i are first calculated using (39), (40) and (35) respectively. The axial strain ϵ_v , the transverse stress σ_t and the shear strain γ_c for the corresponding cracked laminate are calculated using (100) and (104), where use is made of the expressions (91) and (93) applied for the special case $i = j$. Because linear elasticity is assumed it follows that the parameters ϵ_v , σ_t and γ_c are linearly related to the parameters σ , τ , ϵ_c^* , ΔT and Δc , i.e.

$$\epsilon_c = A_1\sigma + B_1\tau + C_1\epsilon_c^* + D_1\Delta T + E_1\Delta c$$

$$\gamma_c = A_2\sigma + B_2\tau + C_2\epsilon_c^* + D_2\Delta T + E_2\Delta c$$

$$\sigma_t = A_3\sigma + B_3\tau + C_3\epsilon_c^* + D_3\Delta T + E_3\Delta c$$

The coefficients in (119-121) are determined numerically by calculating the parameters ϵ_v , γ_c and σ_t for each of the following sets of values for the parameters σ , τ , ϵ_c^* , ΔT and Δc respectively

$$\begin{aligned} &\{1 \ 0 \ 0 \ 0 \ 0\} \\ &\{0 \ 1 \ 0 \ 0 \ 0\} \\ &\{0 \ 0 \ 1 \ 0 \ 0\} \\ &\{0 \ 0 \ 0 \ 1 \ 0\} \\ &\{0 \ 0 \ 0 \ 0 \ 1\}. \end{aligned}$$

On solving (121) for ϵ_c^* and substituting in (119) and (120) it follows that

$$\begin{aligned} \epsilon_c = &\left(A_1 - \frac{C_1}{C_3}A_3\right)\sigma + \frac{C_1}{C_3}\sigma_t + \left(B_1 - \frac{C_1}{C_3}B_3\right)\tau \\ &+ \left(D_1 - \frac{C_1}{C_3}D_3\right)\Delta T + \left(E_1 - \frac{C_1}{C_3}E_3\right)\Delta c, \end{aligned}$$

$$\epsilon_c^* = -\frac{A_3}{C_3}\sigma + \frac{1}{C_3}\sigma_t - \frac{B_3}{C_3}\tau - \frac{D_3}{C_3}\Delta T - \frac{E_3}{C_3}\Delta c,$$

$$\gamma_c = \left(A_2 - \frac{C_2}{C_3} A_3 \right) \sigma + \frac{C_2}{C_3} \sigma_t + \left(B_2 - \frac{C_2}{C_3} B_3 \right) \tau + \left(D_2 - \frac{C_2}{C_3} D_3 \right) \Delta T + \left(E_2 - \frac{C_2}{C_3} E_3 \right) \Delta c . \quad (124)$$

It is expected that the matrix formed by the coefficients of the stress terms on the R.H.S.s of (122-124) will be symmetric. Solutions confirm that such symmetry is valid since numerically it can be shown that $A_2 = B_1$, $A_3 = -C_1$, $B_3 = -C_2$. The relations (122-124) are now written in the form

$$\varepsilon_c = \frac{\sigma}{E_A^c} - \frac{\nu_A^c}{E_A^c} \sigma_t - \frac{\lambda_A^c}{E_A^c} \tau + \alpha_A^c \Delta T + \beta_A^c \Delta c , \quad (125)$$

$$\varepsilon_c^* = -\frac{\nu_A^c}{E_A^c} \sigma + \frac{\sigma_t}{E_T^c} - \frac{\lambda_T^c}{E_A^c} \tau + \alpha_T^c \Delta T + \beta_T^c \Delta c , \quad (126)$$

$$\gamma_c = -\frac{\lambda_A^c}{E_A^c} \sigma - \frac{\lambda_T^c}{E_A^c} \sigma_t + \frac{\tau}{\mu_A^c} + \alpha_s^c \Delta T + \beta_s^c \Delta c , \quad (127)$$

where, for specified values of σ , σ_t , τ , ΔT and Δc , the quantities ε_c , ε_c^* and γ_c are respectively the effective axial, transverse and shear strains of the damaged laminate, and where E_A , E_T , ν_A , α_A , α_T , α_s , β_A , β_T and β_s denote the corresponding effective hygro-thermo-elastic constants of the cracked laminate; the superscript c denoting that the hygro-thermo-elastic constants are for a cracked laminate. The parameters λ_A and λ_T are ratios indicating the degree of shear coupling. The parameters α_s and β_s are expansion coefficients governing the amount of shear strain that can arise when the temperature or moisture content is changed.

15. Derivation of inter-relationships

It is now assumed that the cracked laminate is constrained so that $\gamma_c = 0$. It then follows from (127) that

$$\tau = \frac{\mu_A^c}{E_A^c} \left(\lambda_A^c \sigma + \lambda_T^c \sigma_t - E_A^c \alpha_s^c \Delta T - E_A^c \beta_s^c \Delta c \right) . \quad (128)$$

Substitution in (125) and (126) then leads to the following reduced stress-strain equations that are of the same form as those valid for in-plane biaxial loading modes

$$\varepsilon = \frac{\sigma}{\bar{E}_A^c} - \frac{\nu_A^c}{\bar{E}_A^c} \sigma_t + \alpha_A^c \Delta T + \beta_A^c \Delta c , \quad (129)$$

$$\epsilon_c^* = - \frac{\bar{v}_A^c}{\bar{E}_A^c} \sigma + \frac{\sigma_t}{\bar{E}_T^c} + \alpha_T^c \Delta T + \beta_T^c \Delta c , \quad (130)$$

where

$$\frac{1}{\bar{E}_A^c} = \frac{1}{E_A^c} \left(1 - (\lambda_A^c)^2 \frac{\mu_A^c}{E_A^c} \right) \quad (131)$$

$$\frac{\bar{v}_A^c}{\bar{E}_A^c} = \frac{1}{E_A^c} \left(v_A^c + \lambda_A^c \lambda_T^c \frac{\mu_A^c}{E_A^c} \right) \quad (132)$$

$$\frac{1}{\bar{E}_T^c} = \frac{1}{E_T^c} \left(1 - (\lambda_T^c)^2 \frac{E_T^c \mu_A^c}{(E_A^c)^2} \right) \quad (133)$$

$$\alpha_A^c = \alpha_A^c + \lambda_A^c \frac{\mu_A^c}{E_A^c} \alpha_s^c , \quad (134)$$

$$\alpha_T^c = \alpha_T^c + \lambda_T^c \frac{\mu_A^c}{E_A^c} \alpha_s^c . \quad (135)$$

Now for the special case when the laminate is undamaged, the stress-strain relations (129) and (130) may be written

$$\epsilon = \frac{\sigma}{\bar{E}_A} - \frac{\bar{v}_A}{\bar{E}_A} \sigma_t + \alpha_A \Delta T + \beta_A \Delta c , \quad (136)$$

$$\epsilon_c^* = - \frac{\bar{v}_A}{\bar{E}_A} \sigma + \frac{\sigma_t}{\bar{E}_T} + \alpha_T \Delta T + \beta_T \Delta c , \quad (137)$$

where

$$\frac{1}{\bar{E}_A} = \frac{1}{E_A} \left(1 - (\lambda_A)^2 \frac{\mu_A}{E_A} \right) \quad (138)$$

$$\frac{\bar{\nu}_A}{\bar{E}_A} = \frac{1}{E_A} \left(\nu_A + \lambda_A \lambda_T \frac{\mu_A}{E_A} \right) \quad (139)$$

$$\frac{1}{\bar{E}_T} = \frac{1}{E_T} \left(1 - (\lambda_T)^2 \frac{E_T \mu_A}{(E_A)^2} \right) \quad (140)$$

$$\bar{\alpha}_A = \alpha_A + \lambda_A \frac{\mu_A}{E_A} \alpha_s, \quad (141)$$

$$\bar{\alpha}_T = \alpha_T + \lambda_T \frac{\mu_A}{E_A} \alpha_s. \quad (142)$$

Since $\gamma = 0$, the values σ_o , σ_o^* and τ_o of the applied stresses for which the stress components σ_{yy} and σ_{yz} are zero in the 90° ply of an uncracked laminate are easily found from the relations (39) and (40) by first setting $\sigma_m = \tau_m = 0$, where m is the label for any one of the 90° plies. The resulting simultaneous equations are then solved for the corresponding strains $\epsilon = \epsilon_o$ and $\epsilon^* = \epsilon_o^*$. On using these values together with $\gamma = 0$, the layer stresses σ_i , σ_i^* and τ_i , $i = 1, 2$ are then calculated and used to calculate the corresponding applied stresses σ_o , σ_o^* and τ_o . It follows from (136) and (137) that the strains ϵ_o and ϵ_o^* and the corresponding stresses σ_o , σ_o^* and τ_o satisfy the following relations

$$\epsilon_o = \frac{\sigma_o}{\bar{E}_A} - \frac{\bar{\nu}_A}{\bar{E}_A} \sigma_o^* + \bar{\alpha}_A \Delta T + \bar{\beta}_A \Delta c, \quad (143)$$

$$\epsilon_o^* = - \frac{\bar{\nu}_A}{\bar{E}_A} \sigma_o + \frac{\sigma_o^*}{\bar{E}_T} + \bar{\alpha}_T \Delta T + \bar{\beta}_T \Delta c. \quad (144)$$

Because $\gamma = 0$, these applied stresses also lead to the situation where the cracks in the 90° ply of a damaged laminate are just closed, i.e. the crack surfaces are just touching without any displacement discontinuities across the cracks, and they do not transmit any stress. Thus, from (129) and (130) the strains ϵ_o and ϵ_o^* and the corresponding stresses σ_o , σ_o^* and τ_o also satisfy the following relations

$$\varepsilon_o = \frac{\sigma_o}{\bar{E}_A^c} - \frac{v_A^c}{\bar{E}_A^c} \sigma_o^* + \alpha_A^c \Delta T + \beta_A^c \Delta c ,$$

$$\varepsilon_o^* = - \frac{v_A^c}{\bar{E}_A^c} \sigma_o + \frac{\sigma_o^*}{\bar{E}_T^c} + \alpha_T^c \Delta T + \beta_T^c \Delta c$$

On subtracting (145) from (143) and (146) from (144), in order to eliminate ε_o and ε_o^* , it follows that the following relations must be satisfied for any state of cracking in the 90° plies

$$\left(\frac{1}{\bar{E}_A^c} - \frac{1}{\bar{E}_A} \right) \sigma_o - \frac{v_A^c}{\bar{E}_A^c} - \frac{v_A}{\bar{E}_A} \sigma_o^* + (\alpha_A^c - \alpha_A) \Delta T + (\beta_A^c - \beta_A) \Delta c = 0 ,$$

$$\left(\frac{1}{\bar{E}_A^c} - \frac{1}{\bar{E}_A} \right) \sigma_o + \left(\frac{1}{\bar{E}_T^c} - \frac{1}{\bar{E}_T} \right) \sigma_o^* + (\alpha_T^c - \alpha_T) \Delta T + (\beta_T^c - \beta_T) \Delta c = 0 .$$

The relations can be satisfied for any state of cracking in the 90° plies only if the following five inter-relationships (four of which are independent) are satisfied

$$\frac{\frac{v_A}{\bar{E}_A} - \frac{v_A^c}{\bar{E}_A^c}}{\frac{1}{\bar{E}_A^c} - \frac{1}{\bar{E}_A}} = k , \quad \frac{\frac{1}{\bar{E}_T^c} - \frac{1}{\bar{E}_T}}{\frac{v_A}{\bar{E}_A} - \frac{v_A^c}{\bar{E}_A^c}} = k ,$$

(149)

$$\frac{\frac{1}{\bar{E}_T^c} - \frac{1}{\bar{E}_T}}{\frac{1}{\bar{E}_A^c} - \frac{1}{\bar{E}_A}} = k^2 , \quad \frac{\alpha_T^c - \alpha_T}{\alpha_A^c - \alpha_A} = k , \quad \frac{\beta_T^c - \beta_T}{\beta_A^c - \beta_A} = k$$

where k is a constant (derived in [19] for the case when moisture expansion terms are negligible) that depends only on the geometry and properties of the plies in the laminate.

16. Preliminary results and discussion

In order to test the methodology described in this report, a quasi-isotropic carbon fibre (HTA 5131) reinforced epoxy (Ciba Fiberdux F922) laminate of the form [-45/45/0/90], having a single central 90° ply is considered. All plies in the laminate have thickness

0.125 mm, and have the following properties that were used when describing a generalised framework to predict progressive ply crack formation [20] :

$$\begin{array}{lll} E_A = 136.6 \text{ GPa} & E_T = 9.79 \text{ GPa} & \nu_A = 0.286 \\ \nu_T = 0.455 & \mu_A = 6.474 \text{ GPa} & \\ \alpha_A = -0.458 \times 10^{-6} / ^\circ\text{C} & \alpha_T = 38.2 \times 10^{-6} / ^\circ\text{C} . & \end{array}$$

Moisture expansion coefficients are neglected as their effect is analogous to that of the thermal expansion coefficients. Predictions have been made of the hygro-thermo-elastic constants appearing in (125-127) for a range of crack separations defining crack densities in the range 0 to 4 cracks/mm. Because the stress transfer model has been designed for any number of plies it is possible to improve the accuracy of predictions using ply refinement techniques by representing each ply in the laminate by a set of n elemental layers having the same thickness and properties. In such a way the effects of through-thickness variations in the axial stress distribution can be assessed. Figs. 2-5 show the results of such calculations for those parameters that exhibit a significant variation with respect to both the crack density and the parameter n defining ply refinement. It is seen that by choosing $n = 1$ lower bound predictions for the thermo-elastic constants are obtained. As n increases the predicted values increase and appear to converge sufficiently for practical purposes when $n = 5$. It should be noted that the axial thermal expansion coefficient shown in Fig.5 is the most sensitive to the presence of cracks in the 90° ply. The transverse Young's modulus and transverse thermal expansion coefficient hardly vary with crack density as expected, and have not been shown. The shear coupling constants λ_A , λ_T and α_s are negligible for the quasi-isotropic laminate being discussed. These parameters have significant values in the case of $[\theta/90]_s$ laminates as will now be shown for the case $\theta = 30^\circ$.

A glass fibre reinforced epoxy laminate of the form $[30/90]_s$ is now considered where 30-plyies and the 90-plyies have the same thickness 0.25 mm, and have the following properties typical of a unidirectional Silenka E-glass 1200tex fibre reinforced MY750/HY917/DY063 epoxy composite [21]:

$$\begin{array}{lll} E_A = 45.6 \text{ GPa} & E_T = 16.2 \text{ GPa} & \nu_A = 0.278 \\ \nu_T = 0.4 & \mu_A = 5.83 \text{ GPa} & \\ \alpha_A = 8.6 \times 10^{-6} / ^\circ\text{C} & \alpha_T = 26.4 \times 10^{-6} / ^\circ\text{C} . & \end{array}$$

Predictions have again been made of the hygro-thermo-elastic constants of cracked laminates appearing in (125-127) for a range of crack separations defining crack densities in the range 0 to 4 cracks/mm. Figs.6-12 show the results of such calculations for those parameters that exhibit a significant variation with respect to both the crack density and the parameter n defining ply refinement. It is seen that by choosing $n = 1$, and considering absolute values, lower bound predictions for the thermo-elastic constants are obtained with the exception of results shown in Fig.12. As n increases the predicted values appear to converge sufficiently for practical purposes when $n = 5$. For the GRP $[30/90]_s$ laminate the selection of $n = 1$ leads to predictions that deviate significantly from those for $n = 5$, indicating that ply refinement is essential if accurate estimates are required. The transverse Young's modulus and transverse thermal expansion coefficient hardly vary with crack density as expected, and again have not been shown.

The predicted values of the hygro-thermo-elastic constants appearing in (125-127) have been used to check whether the inter-relationships (149) derived in section 15 are satisfied. The value of the constant $k = -0.00626$ for the carbon fibre epoxy laminate and $k = -0.01827$ for the glass fibre epoxy laminate. Such checks have been built into the software system that predicts the values of the hygro-thermo-elastic constants. In all cases considered the inter-relationships have been shown to be satisfied within numerical error. The quality of solutions has also been checked by examining the satisfaction of the differential equations (88) and (89),

and boundary conditions imposed, including the satisfaction of the interface conditions for both tractions and displacements especially in regions close to the ply crack tips.

It is emphasised that highly accurate predictions of the hygro-thermo-elastic constants of cracked laminates, and of the stress and displacement distributions at all points within the laminate, can be made using the methods described in this report. The predictions are expected to be superior to those that would result from conventional finite element analysis as the boundary and interface conditions can be satisfied more accurately. In contrast to finite element analysis, where it is not possible to guarantee the continuity of both tractions and displacements at element boundaries, the methodology described here can guarantee such continuity. Indeed one can think of ply refinement as a finite super-element technique where the elements are in the form of layers such that there is continuity of both tractions and displacements between the layers.

17. Conclusions

- 1) By assuming that the shear stresses σ_{xy} and σ_{xz} are piece-wise linear functions of the through-thickness coordinate, it is possible to derive an analytical representation for the stress and displacement fields for any general symmetric laminate containing uniform distributions of cracks in the 90° plies that satisfy exactly the equilibrium equations, the compatibility equations, four stress-strain relations, and the interface continuity conditions. By satisfying the remaining two stress-strain relations on averaging them through the thickness of each ply or elemental layer, the stress transfer problem is reduced to solving a system of ordinary differential equations that can be solved by standard techniques.
- 2) The stress and displacement solutions can be used to derive the values of all hygro-thermo-elastic constants of a uniformly cracked laminate. Furthermore, it has been confirmed from predictions that the values of these constants satisfy various inter-relationships defined by (149) that have been derived by considering ply crack closure.
- 3) Ply refinement techniques can be used to obtain more accurate predictions of the hygro-thermo-elastic constants of cracked laminates. The sub-division of plies into five elemental layers of equal thickness leads to sufficiently accurate predictions for quasi-isotropic $[-45/45/0/90]_s$ carbon fibre and $[30/90]_s$ glass fibre epoxy laminates, and for other types of laminate that have not been discussed. Inaccurate predictions have been shown to result (especially for the GRP laminate) if the stress transfer model is applied without ply refinement, implying the importance of the through-thickness variations of the stress and displacement distribution.
- 4) The methodology presented, when used in conjunction with ply refinement techniques, can be regarded as a finite super element technique where accurate solutions can be obtained for which continuity of both tractions and displacement fields at all elemental layer interfaces can be guaranteed.

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APPENDIX A

Expressions are first derived for the stress components referred to local coordinates in terms of the stress components for global coordinates as follows for the i^{th} layer

$$\sigma_{xx}^{i'} = \sigma_{xx}^i ,$$

$$\sigma_{yy}^{i'} = m_i^2 \sigma_{yy}^i + n_i^2 \sigma_{zz}^i - 2m_i n_i \sigma_{yz}^i ,$$

$$\sigma_{zz}^{i'} = n_i^2 \sigma_{yy}^i + m_i^2 \sigma_{zz}^i + 2m_i n_i \sigma_{yz}^i ,$$

$$\sigma_{xy}^{i'} = m_i \sigma_{xy}^i - n_i \sigma_{xz}^i ,$$

$$\sigma_{xz}^{i'} = n_i \sigma_{xy}^i + m_i \sigma_{xz}^i ,$$

$$\sigma_{yz}^{i'} = m_i n_i \sigma_{yy}^i - m_i n_i \sigma_{zz}^i + (m_i^2 - n_i^2) \sigma_{yz}^i ,$$

where $m_i = \cos\phi_i$ and $n_i = \sin\phi_i$, where ϕ_i is the angle (measured clockwise) between the global y-axis and the fibre direction of the i^{th} layer.

The strain components referred to local coordinates can be expressed in terms of global stress components by making use of the following stress-strain relations for the i^{th} layer expressed in terms of stress and strain components referred to local coordinates

$$\epsilon_{xx}^{i'} = \frac{1}{E_t^i} \sigma_{xx}^{i'} - \frac{\nu_a^i}{E_A^i} \sigma_{yy}^{i'} - \frac{\nu_t^i}{E_T^i} \sigma_{zz}^{i'} + \alpha_t^i \Delta T + \beta_t^i \Delta c ,$$

$$\epsilon_{yy}^{i'} = -\frac{\nu_a^i}{E_A^i} \sigma_{xx}^{i'} + \frac{1}{E_A^i} \sigma_{yy}^{i'} - \frac{\nu_A^i}{E_A^i} \sigma_{zz}^{i'} + \alpha_A^i \Delta T + \beta_A^i \Delta c ,$$

$$\epsilon_{zz}^{i'} = -\frac{\nu_t^i}{E_T^i} \sigma_{xx}^{i'} - \frac{\nu_A^i}{E_A^i} \sigma_{yy}^{i'} + \frac{1}{E_T^i} \sigma_{zz}^{i'} + \alpha_T^i \Delta T + \beta_T^i \Delta c ,$$

$$\epsilon_{xy}^{i'} = \frac{1}{c_{xy}^i} \sigma_{xy}^{i'} , \quad (A10)$$

$$\epsilon_{xz}^{i'} = \frac{1}{2\mu_1^i} \sigma_{xz}^{i'} , \quad (A11)$$

$$\epsilon_{yz}^{i'} = \frac{1}{2\mu_A^i} \sigma_{yz}^{i'} . \quad (A12)$$

Finally the global strain components may be expressed in terms of global stress components by substituting (A7-A12) in the following relations

$$\epsilon_{xx}^i = \epsilon_{xx}^{i'} , \quad (A13)$$

$$\epsilon_{yy}^i = m_1^2 \epsilon_{yy}^{i'} + n_1^2 \epsilon_{zz}^{i'} + 2m_1 n_1 \epsilon_{yz}^{i'} , \quad (A14)$$

$$\epsilon_{zz}^i = n_1^2 \epsilon_{yy}^{i'} + m_1^2 \epsilon_{zz}^{i'} - 2m_1 n_1 \epsilon_{yz}^{i'} , \quad (A15)$$

$$\epsilon_{xy}^i = m_1 \epsilon_{xy}^{i'} + n_1 \epsilon_{xz}^{i'} , \quad (A16)$$

$$\epsilon_{xz}^i = - n_1 \epsilon_{xy}^{i'} + m_1 \epsilon_{xz}^{i'} , \quad (A17)$$

$$\epsilon_{yz}^i = - m_1 n_1 \epsilon_{yy}^{i'} + m_1 n_1 \epsilon_{zz}^{i'} + (m_1^2 - n_1^2) \epsilon_{yz}^{i'} . \quad (A18)$$

The resulting stress-strain equations relating global strain components to global stress components are given by (4-9) where the hygro-thermo-elastic constants are specified by the relations (10-26).

APPENDIX B

The inverse form of the relations (41-43) is given by

$$\varepsilon = A'\sigma + B'\sigma^* + D'\tau + P'\Delta T + U'\Delta c$$

$$\varepsilon^* = B'\sigma + C'\sigma^* + E'\tau + Q'\Delta T + V'\Delta c$$

$$\gamma = D'\sigma + E'\sigma^* + F'\tau + R'\Delta T + W'\Delta c ,$$

where methods of linear algebra can be used to relate the primed constants to the parameters defined by (44-57).

In order to introduce the more familiar hygro-thermo-elastic constants of a laminate the relations (B1-B3) are now written

$$\varepsilon = \frac{\sigma}{E_A} - \frac{\nu_A}{E_A} \sigma^* - \frac{\lambda_A}{E_A} \tau + \alpha_A \Delta T + \beta_A \Delta c$$

$$\varepsilon^* = -\frac{\nu_A}{E_A} \sigma + \frac{\sigma^*}{E_T} - \frac{\lambda_T}{E_A} \tau + \alpha_T \Delta T + \beta_T \Delta c ,$$

$$\gamma = -\frac{\lambda_A}{E_A} \sigma - \frac{\lambda_T}{E_A} \sigma^* + \frac{\tau}{\mu_A} + \alpha_s \Delta T + \beta_s \Delta c$$

where E_A , E_T , ν_A , μ_A , α_A , α_T , β_A and β_T are the Young's moduli, Poisson's ratio, shear modulus, thermal and moisture expansion coefficients for the undamaged multiple-ply laminate. The parameters λ_A and λ_T are ratios associated with the effect of shear stress on axial and transverse strain respectively. The parameters α_s and β_s are the thermal and moisture coefficients associated respectively with the effects of temperature and moisture changes on the shear strain.

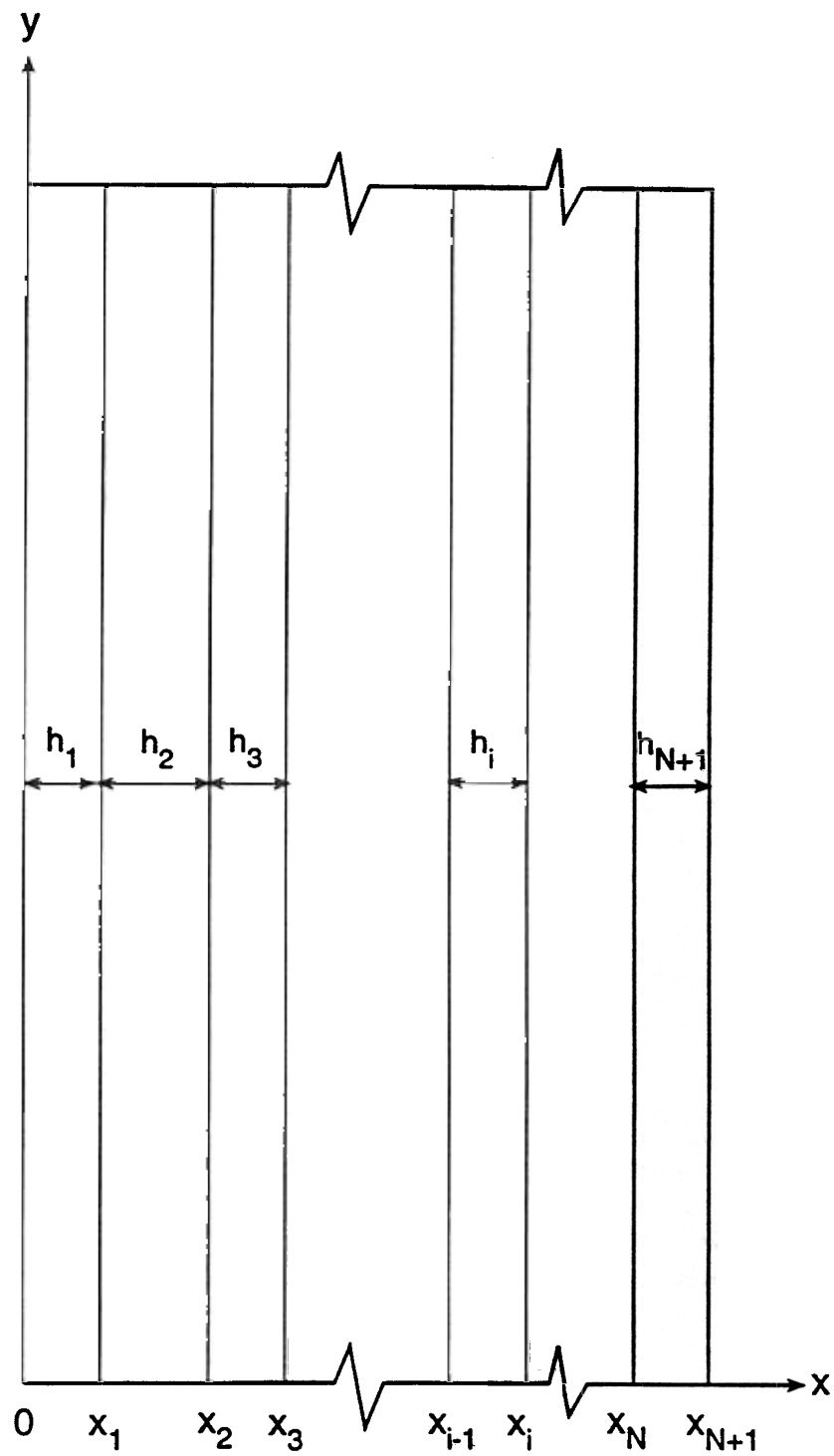


FIG.1: Schematic edge view of the right hand quadrant of a general symmetric laminate indicating coordinate axes and the geometrical interpretation of the layer parameters h_i and $x_i, i = 1...N+1$.

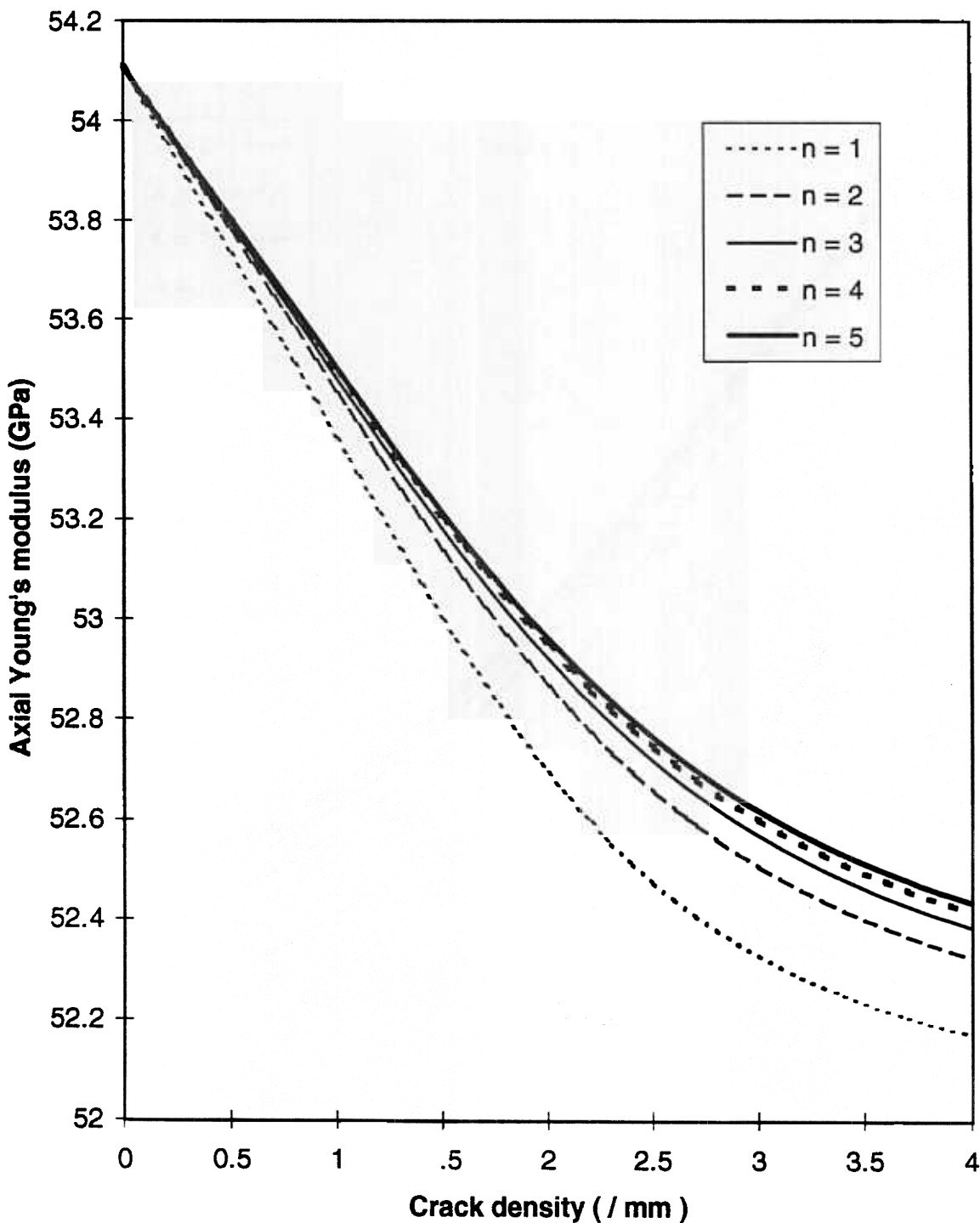


FIG.2: Dependence of the axial Young's modulus E_A of a $[-45/45/0/90]$, carbon fibre epoxy laminate on the density of cracks in the 90° ply.

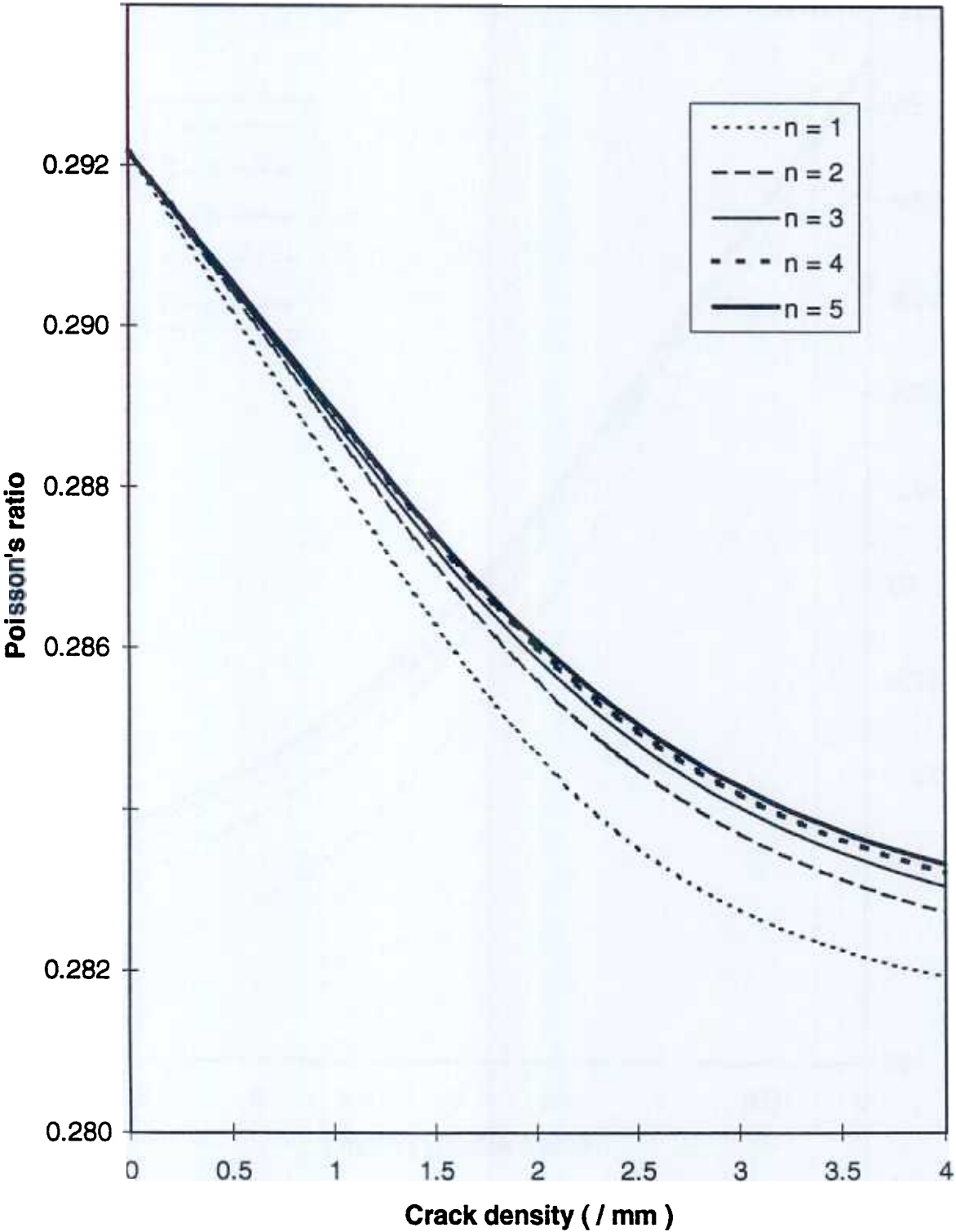


FIG.3: Dependence of the axial Poisson's ratio ν_A of a $[-45/45/0/90]_s$ carbon fibre epoxy laminate on the density of cracks in the 90° ply.

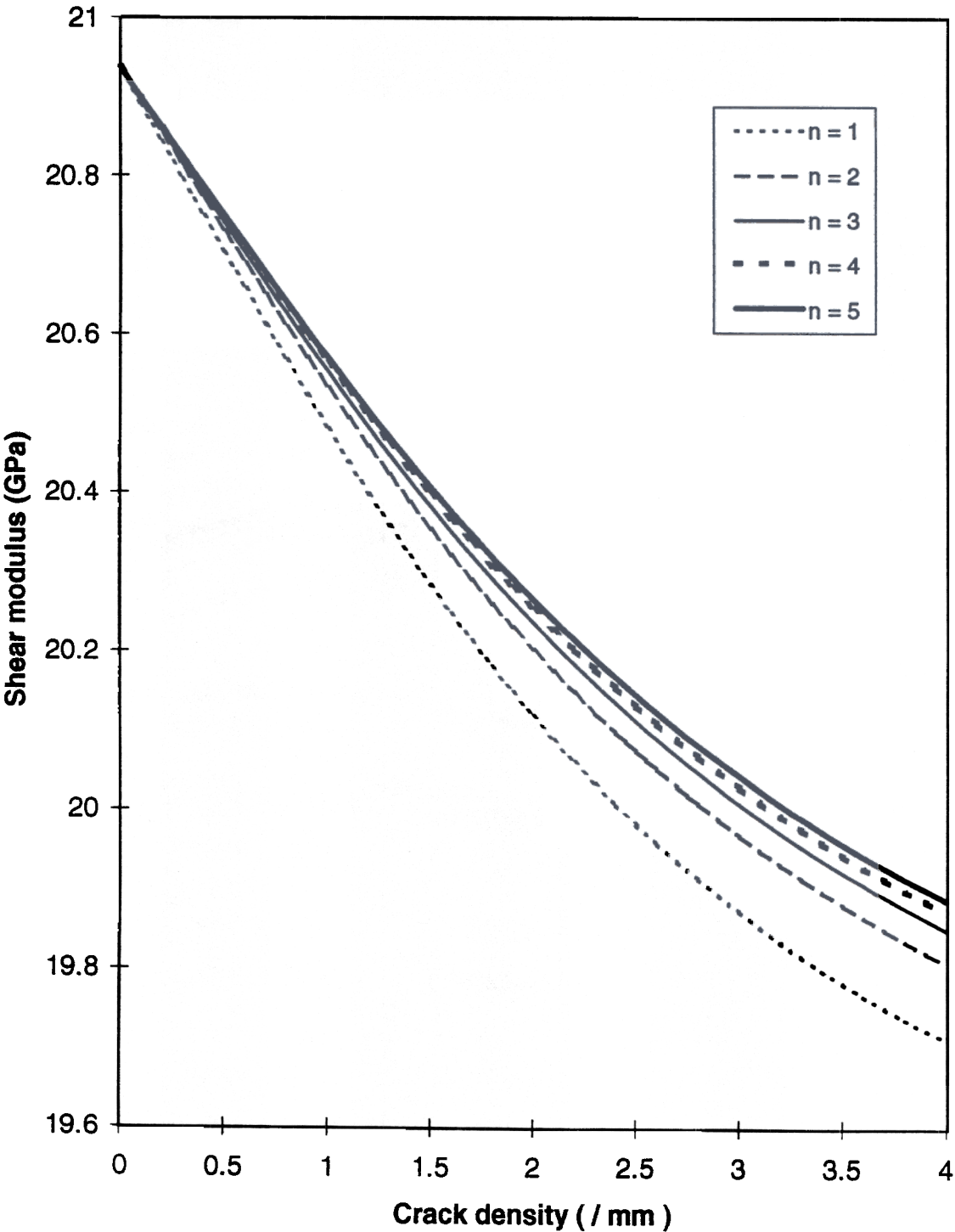


FIG.4: Dependence of the axial shear modulus μ_A of a $[-45/45/0/90]_s$ carbon fibre epoxy laminate on the density of cracks in the 90° ply.

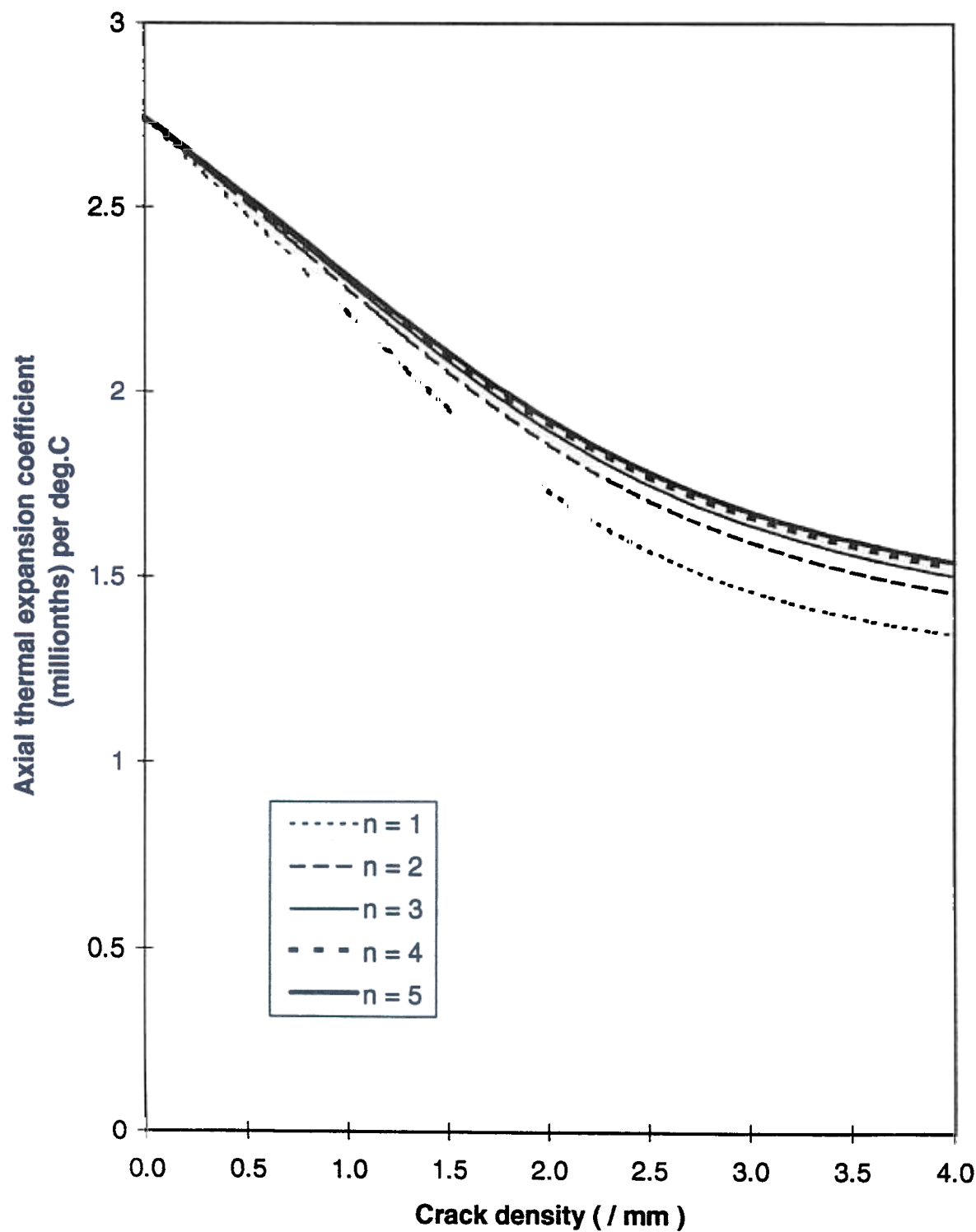


FIG.5: Dependence of the axial thermal expansion coefficient α_A of a $[-45/45/0/90]_s$ carbon fibre epoxy laminate on the density of cracks in the 90° ply.

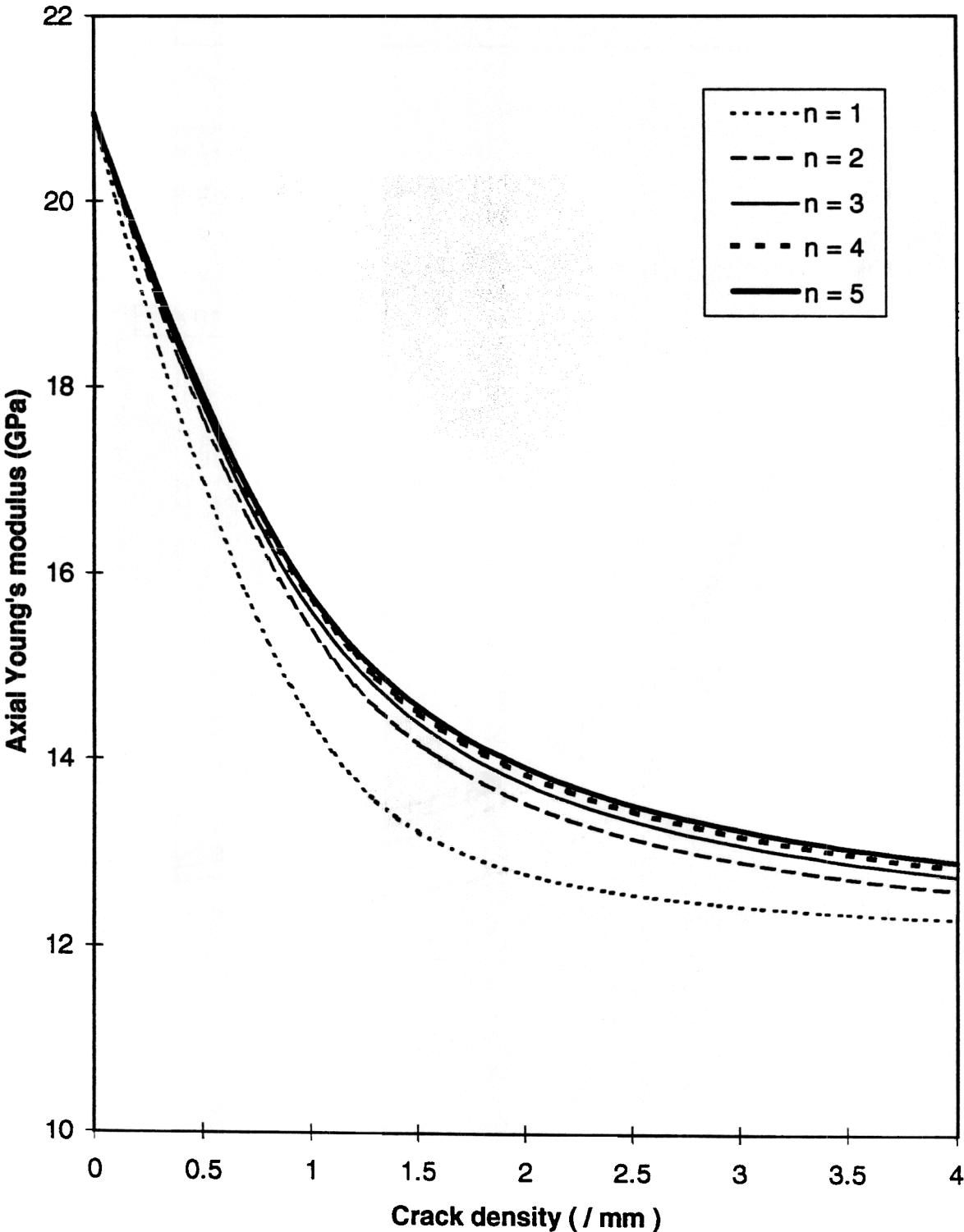


FIG.6: Dependence of the axial Young's modulus E_A of a $[30/90]_s$ glass fibre epoxy laminate on the density of cracks in the 90° ply.

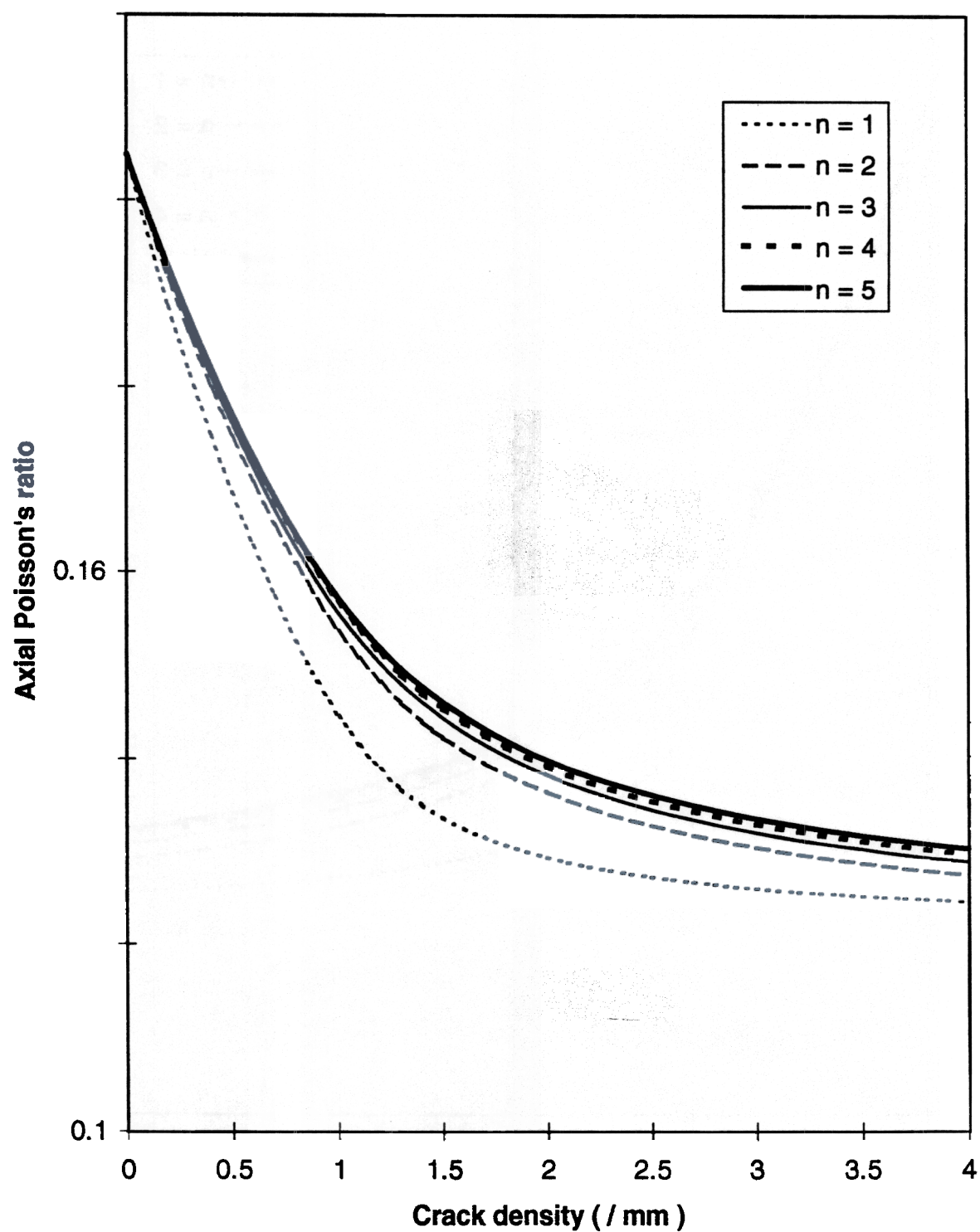


FIG.7: Dependence of the axial Poisson's ratio ν_A of a $[30/90]_s$ glass fibre epoxy laminate on the density of cracks in the 90° ply.

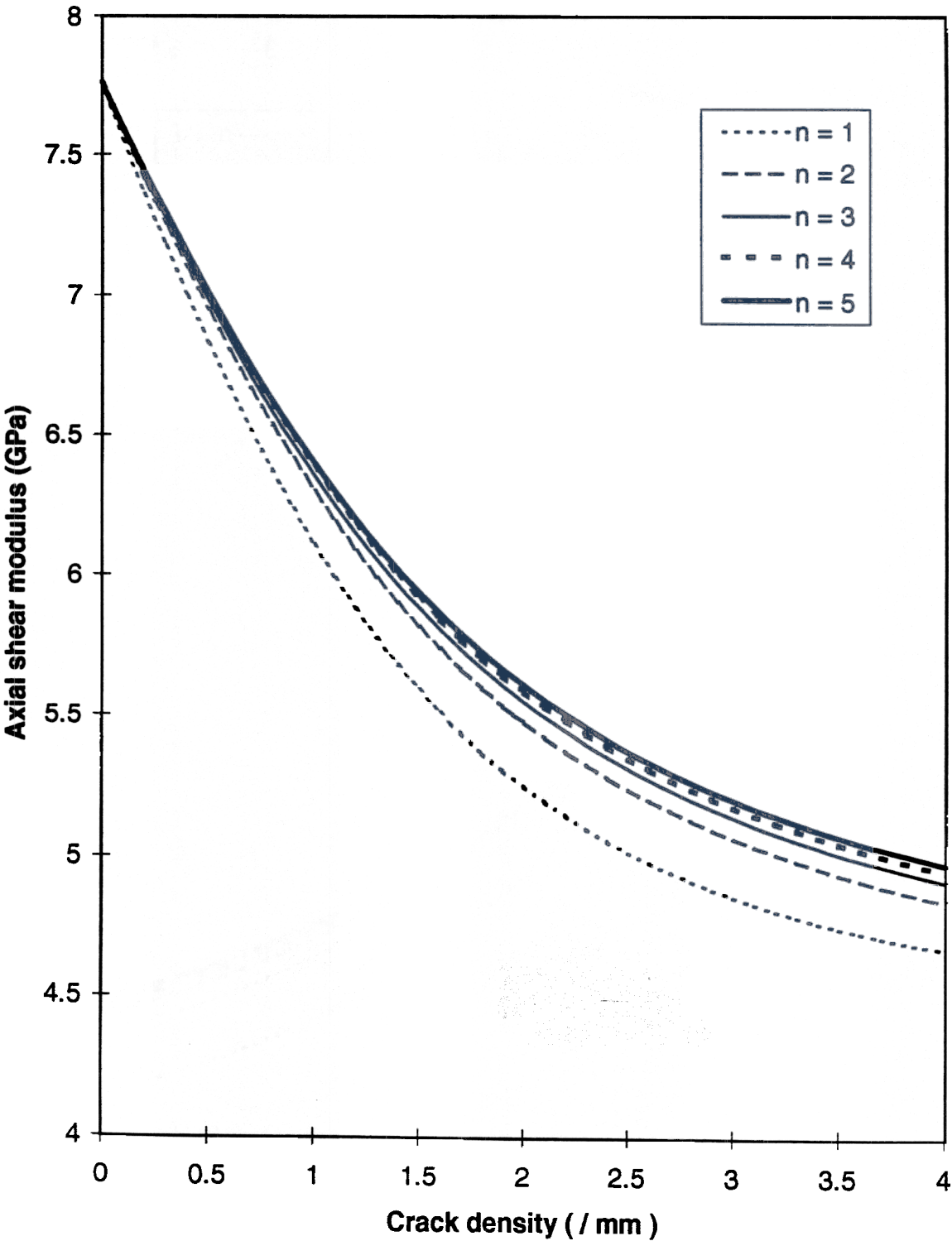


FIG.8: Dependence of the axial shear modulus μ_A of a $[30/90]_s$ glass fibre epoxy laminate on the density of cracks in the 90° ply.

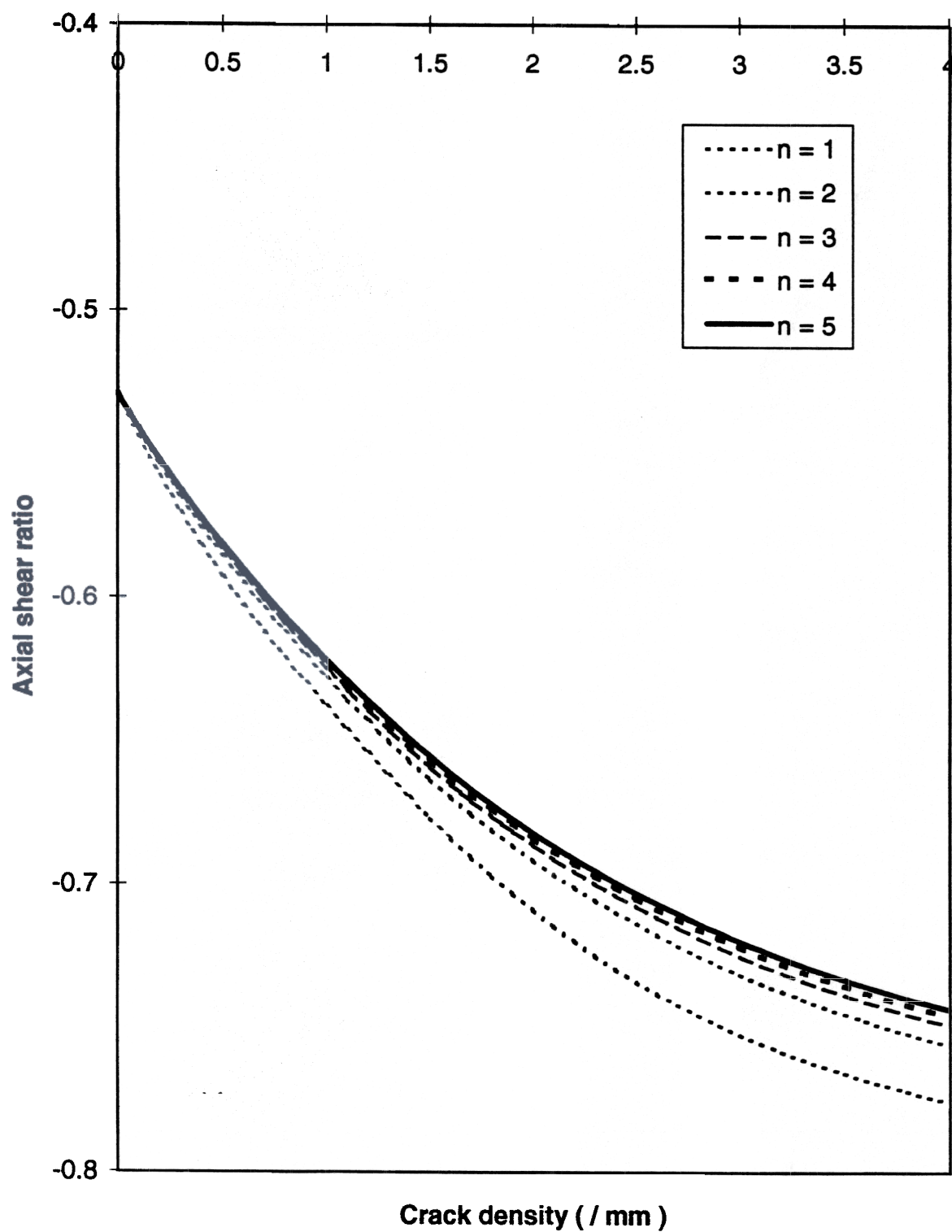


FIG.9: Dependence of the axial shear ratio λ_A of a $[30/90]_s$ glass fibre epoxy laminate on the density of cracks in the 90° ply.

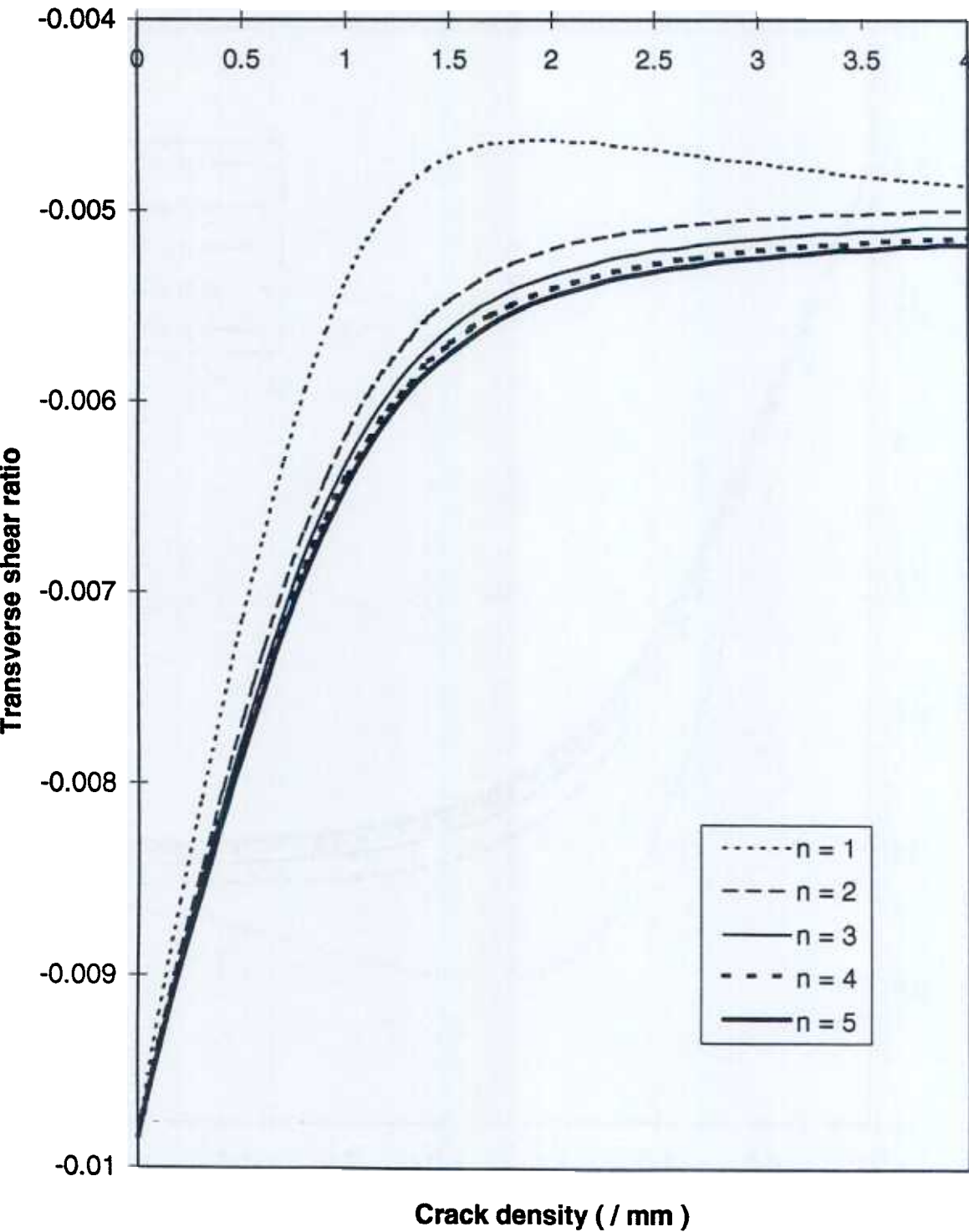


FIG.10: Dependence of the transverse shear ratio λ_r of a $[30/90]_s$ glass fibre epoxy laminate on the density of cracks in the 90° ply.

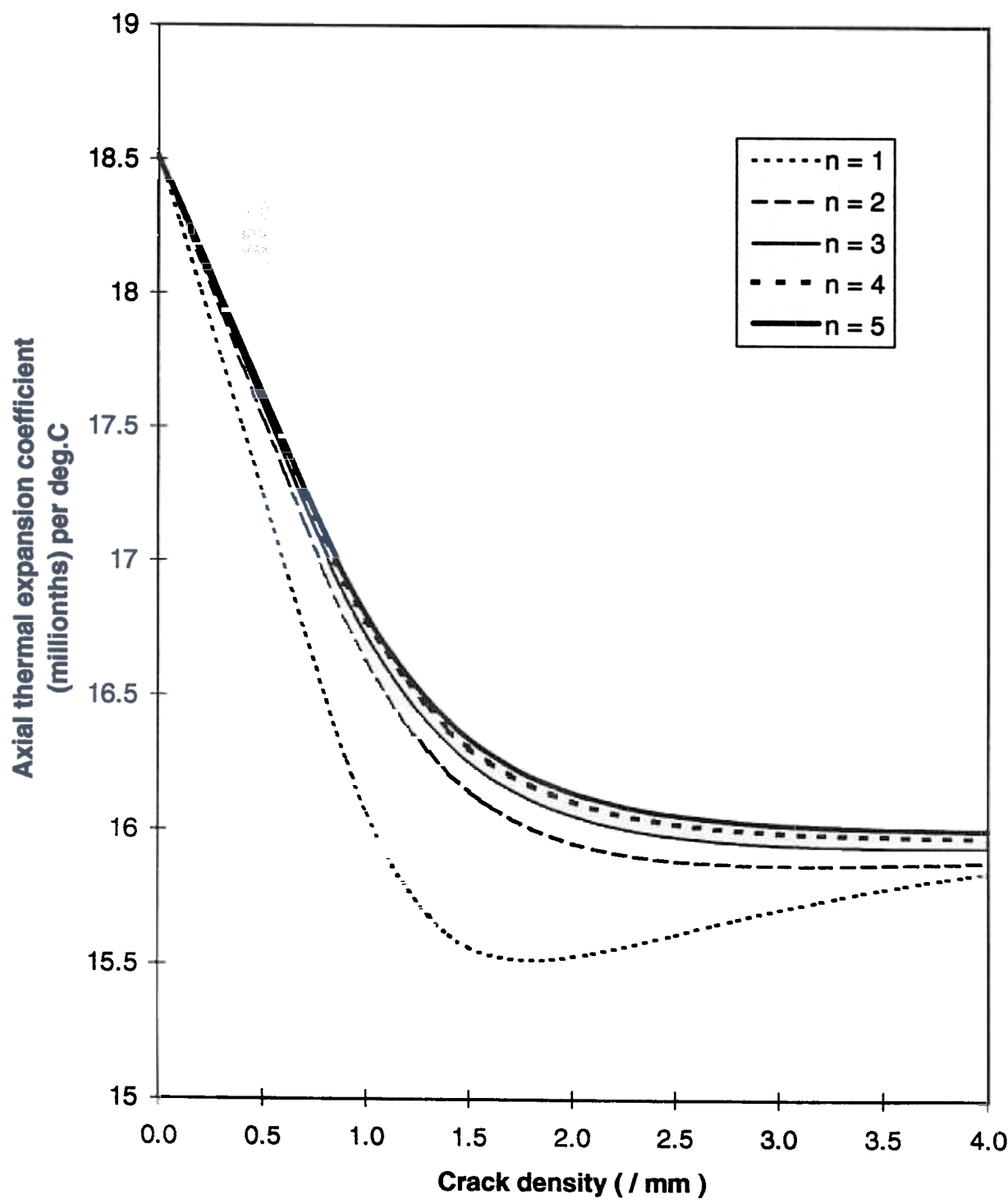


FIG.11: Dependence of the axial thermal expansion coefficient α_A of a $[30/90]_s$ glass fibre epoxy laminate on the density of cracks in the 90° ply.

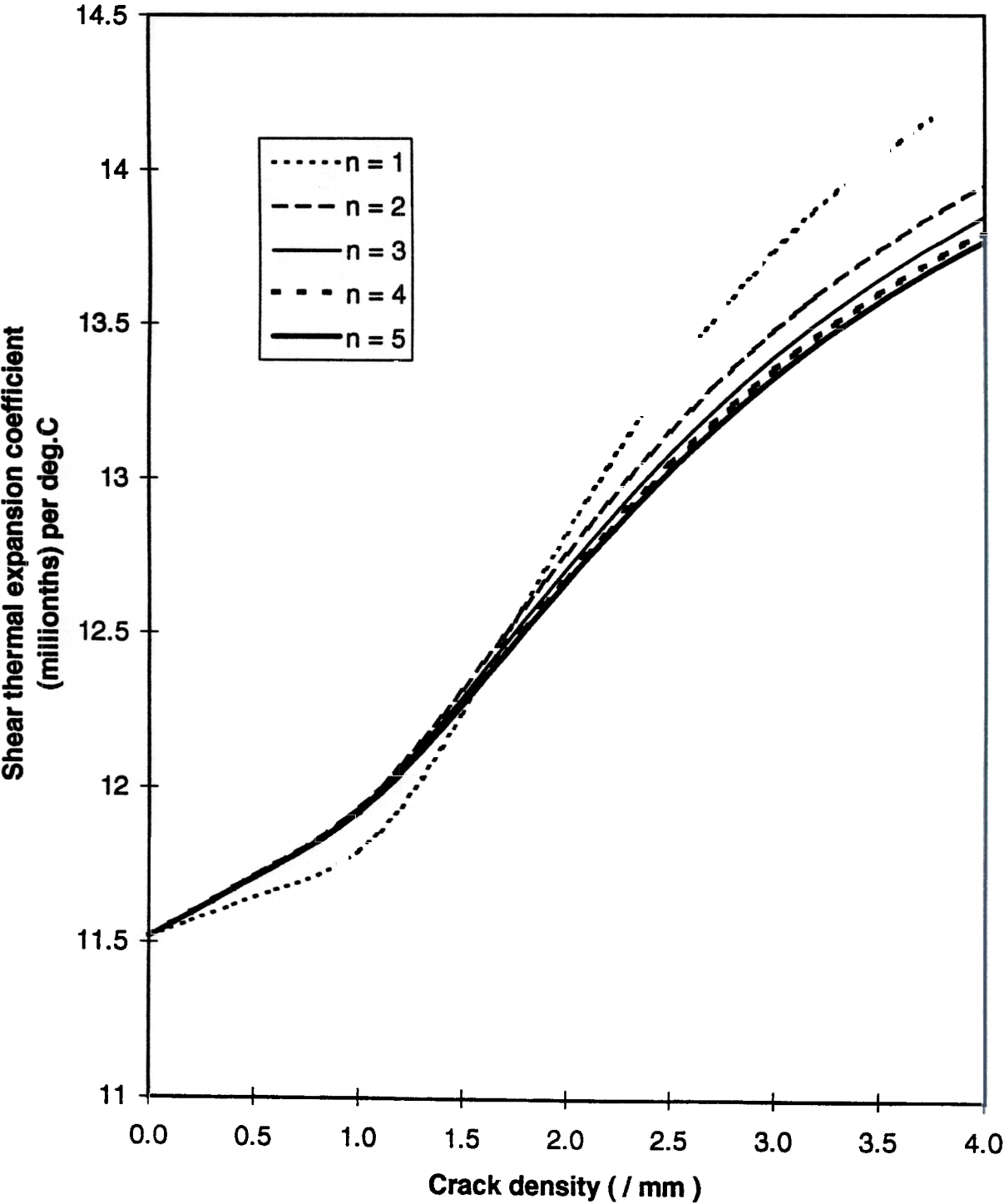


FIG.12: Dependence of the shear thermal expansion coefficient α_s of a $[30/90]_s$ glass fibre epoxy laminate on the density of cracks in the 90° ply.