

**Hybrid Methods: Review of particle-based numerical
methods and their coupling to other continuum methods**

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November 2010

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ABSTRACT

This report reviews the current state of meshless methods, summarises some commonly-used examples, and describes methods for coupling meshless methods to finite element methods. The aim of the report is to provide an explanation of why meshless methods are useful and to identify which meshless methods and coupling methods should be investigated further.

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Contents

1	Introduction	1
1.1	Document structure	1
1.2	Why use meshless methods?	1
1.3	Definitions and notation	2
2	Meshless Methods	3
2.1	Smoothed particle hydrodynamics	4
2.1.1	Drawbacks of SPH	5
2.2	Reproducing Kernel Particle Methods	6
2.2.1	Drawbacks of RKPM	7
2.3	Element-free Galerkin methods	7
2.3.1	Drawbacks of EFG methods	9
2.4	Material point methods	9
2.4.1	Drawbacks of MPM	10
2.5	Application of the essential boundary condition	10
2.6	Software	11
3	Coupling between meshless methods and FE	12
3.1	Master-slave coupling	13
3.2	Coupling via ramp functions	13
3.3	Coupling with reproducing conditions	14
3.4	Bridging domain methods	15
3.5	Coupling with Lagrange multipliers	16
3.6	Criteria for judging coupling methods	16
4	Recommendations for further work	17
5	Literature review	18
5.1	Review papers	18
5.2	Applications	20

1 Introduction

This report reviews the current state of meshless methods, summarises some commonly-used examples, and describes methods for coupling meshless methods to finite element methods. The aim of the report is to provide an explanation of why meshless methods are useful and to identify which meshless methods and coupling methods should be investigated further.

1.1 Document structure

The document consists of four sections. The introduction describes the aim of the document, gives reasons why meshless methods are useful, and defines some terms used in the rest of the report. The second section describes the formulation of some commonly-used methods, addresses their drawbacks, and lists some software implementations of meshless methods. The third section discusses methods for coupling meshless methods to finite element methods. The fourth section makes some recommendations for the focus of further work. The fifth section is a summary of the literature that has been read during the preparation of this document, split into review papers and applications of the methods.

1.2 Why use meshless methods?

Finite element (FE) methods have become an increasingly popular tool for simulation of engineering and physics problems over the last thirty years. As cheap computing power has become more widely available, the range of problems that can be solved using FE has increased, but there are still some problems that FE methods do not solve efficiently or accurately.

FE methods are not ideally suited for solution of problems involving a large amount of deformation. FE methods rely on a mesh to generate a numerical solution to a problem. An initial mesh is usually created from evenly shaped triangular and quadrilateral elements in two dimensions, or tetrahedra, triangular prisms, and hexahedra in three dimensions. A local approximation technique is applied within each element, resulting in a large matrix of equations that link the unknown quantities at different points within the mesh. As the model is solved, the mesh deforms and the elements become less evenly shaped. The accuracy and convergence of the numerical solution is dependent on the well-shapedness of the elements, with highly distorted elements leading to near-singular matrices of equations.

One approach to avoiding the problems associated with mesh deformation is remeshing, where a new mesh of regularly shaped elements is created during the solution process.

This approach is computationally expensive and has potential loss of accuracy as it requires the mapping of the solution from one mesh to the next.

Meshless methods avoid the problems associated with mesh deformation by not having a mesh. The methods still use local approximations, centred on a set of points, and discretisation to calculate an approximate solution to the governing equations, but the domains over which each local approximation is applied do not (necessarily) deform with the material. The approximation function associated with each point have a fixed “domain of influence” on which they are non-zero, and the connectivity of a given point depends on which other points lie within that point’s domain of influence at any given time.

FE methods are not always suitable for problems involving rapid change in value of a variable, such as shock waves or other discontinuities, because the mesh required to describe such rapid changes accurately can be too computationally expensive. Meshless methods generally use local approximations that are better able to cope with sharp changes.

1.3 Definitions and notation

Throughout the following, u is the field variable of interest (e.g. displacement, temperature, electrical field, etc.). For convenience it is assumed to be a scalar quantity but it is straightforward to extend all derivations of methods etc. to a vector quantity. $\mathbf{x}$ denotes a position vector. The derivations in this report apply to a space of any dimension.

The **strong form** of a partial differential equation (pde) is the equation stated in the form

$$\nabla \cdot (a(\mathbf{x}, u) \nabla u) + \mathbf{b}(\mathbf{x}, u) \cdot \nabla u + c(\mathbf{x}, u)u + d(\mathbf{x}, u) = 0, \quad \mathbf{x} \in \Omega. \quad (1)$$

This is the form in which most pdes are generally stated. The **weak form** is created by multiplying both sides of the equation by some function $v(\mathbf{x})$ and integrating over the domain Ω to give

$$\int_{\Omega} v(\mathbf{x}) (\nabla \cdot (a(\mathbf{x}, u) \nabla u) + \mathbf{b}(\mathbf{x}, u) \cdot \nabla u + c(\mathbf{x}, u)u + d(\mathbf{x}, u)) d\Omega = 0. \quad (2)$$

The main advantage of this approach is that for most pdes, Green’s theorem can be applied which reduces the order of the derivatives in the equation, meaning that the space of possible solutions is larger (since functions with discontinuous first derivatives are now feasible solutions) and that numerical solution will only require calculation of first-order derivatives (thus reducing computational complexity).

The weak form is true for all functions v , but solution via numerical approximation of the weak form requires the choice of a particular example. The choice of v , known as the **test function** determines the discretised form of the equations. Methods derived from the weak form of the equation are generally known as **Galerkin** methods.

The **essential boundary condition** for a pde where u is the unknown quantity is the condition that specifies values of u on some part of the boundary. Examples include fixed temperature conditions, clamping conditions, and fixed displacements. Applied fluxes and forces are not essential boundary conditions because those conditions specify the derivative of the field variable of interest.

A **complete polynomial basis of order k** for a space of dimension d is $\mathbf{p}(\mathbf{x}) = \mathbf{x}^\alpha : |\alpha| \leq k$, where $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_d^{\alpha_d}$, and $|\alpha| = \sum_{i=1}^d \alpha_i$. For example in two dimensions the complete basis of order 3 is $1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3$. In one dimension, the complete basis of order k is $1, x, x^2, \dots, x^k$. The complete polynomial basis is the set of basis functions of the set of all polynomials on that space of order less than or equal to k . The number of polynomials in the basis is $q = (\prod_{i=1}^d (k + i)) / d!$

2 Meshless Methods

This section describes the derivation of some of the most common meshless methods. All the methods attempt to define an approximation u_h to a function u , based on a size parameter h , a set of discretisation points $\mathbf{x}_i, i = 1, 2, \dots, n$ and an associated set of nodal parameters $u_i, i = 1, 2, \dots, n$. Typically the approximation can be written in the form $u_h(\mathbf{x}) = \sum_{i=1}^n N_i(\mathbf{x})u_i$ where the N_i are called the **shape functions**.

In some cases the methods can be extended to produce approximations of the form

$$u_h(\mathbf{x}) = \sum_{i=1}^n N_i(\mathbf{x}) \left(u_i + \sum_{j=1}^{N_j} v_j M_j(\mathbf{x}) \right) \quad (3)$$

where the u_i and v_j are unknown quantities and the $N_i(\mathbf{x})$ and $M_j(\mathbf{x})$ are known functions. This extension is useful if some details of the likely behaviour of the solution are known. For instance, it is known that displacements in the region of a crack tip have a dependence on $r^{1/2}$, and so a function describing the behaviour could be included in the shape function. An extension of this type leads to the extended finite element method (XFEM).

The solution of a pde using the methods involves applying the pde in its strong or weak forms to the continuous form of the approximation u_h and then discretising to obtain a set of equations, often linear or linearizable, that can then be solved to determine the unknown parameters. Note that unlike FE methods, the parameter associated with a point is not necessarily the value of the unknown quantity at that point.

The differences between the methods depend on the construction method used to create u_h and on the form of the pde to which the method is applied (strong or weak forms).

2.1 Smoothed particle hydrodynamics

Smoothed particle hydrodynamics (SPH) is based on the fact that

$$u(\mathbf{x}) = \int_{\Omega} u(\mathbf{y}) \delta(|\mathbf{x} - \mathbf{y}|) d\Omega_{\mathbf{y}}. \quad (4)$$

The method uses functions $w_h(\mathbf{x})$ that approximate $\delta(\mathbf{x})$ where h is a smoothing parameter and $w_h(\mathbf{x})$ has the following properties:

1. w_h is strictly positive on some subdomain of Ω and is zero outside of that subdomain.
2. $\int_{\Omega} w_h(\mathbf{x}) d\Omega_{\mathbf{x}} = 1$;
3. $w_h(\mathbf{x}) \rightarrow \delta(\mathbf{x})$ as $h \rightarrow 0$;
4. $w_h(\mathbf{x})$ is a monotonically decreasing function of $|\mathbf{x}|$.

Commonly used examples include:

$$w_h^G(\mathbf{x}) = \frac{\exp(-\|\mathbf{x}\|^2/h^2)}{(\pi h^2)^{d/2}}, \quad (5)$$

$$w_h^C(\mathbf{x}) = \frac{C}{h^d} \begin{cases} 1 - 3\|\mathbf{x}\|^2/(2h^2) + 3\|\mathbf{x}\|^3/(4h^3) & 0 \leq \|\mathbf{x}\| \leq h, \\ (2 - \|\mathbf{x}\|/h)^3/4 & h \leq \|\mathbf{x}\| \leq 2h, \\ 0 & \text{otherwise,} \end{cases} \quad (6)$$

$$w_h^Q(\mathbf{x}) = \frac{C}{h^d} \begin{cases} 1 - 6\|\mathbf{x}\|^2/h^2 + 8\|\mathbf{x}\|^3/h^3 - 3\|\mathbf{x}\|^4/h^4 & 0 \leq \|\mathbf{x}\| \leq h, \\ 0 & \text{otherwise,} \end{cases} \quad (7)$$

where d is the dimension of the problem, C is an appropriate normalisation constant such that condition 2 above is satisfied, and w_h^G is the Gaussian function, w_h^C is the cubic spline function, and w_h^Q is the quartic spline function. The kernel function is generally defined as being symmetric, as is the case for all of these examples.

In its discretised form, the approximation is written

$$u_h(\mathbf{x}) = \sum_{i=1}^n w_h(\mathbf{x} - \mathbf{x}_i) \Delta V_i u_i, \quad (8)$$

or, defining $N_i(\mathbf{x}) = w_h(\mathbf{x} - \mathbf{x}_i) \Delta V_i$,

$$u_h(\mathbf{x}) = \sum_{i=1}^n N_i(\mathbf{x}) u_i. \quad (9)$$

SPH methods are usually applied directly to the strong form of the governing pde, and so their implementation requires calculation of the second derivative of the shape functions N_i . There are various approaches to gradient calculation that allow the user to enforce the conservation laws and symmetry that are frequently a feature of the governing equations. These approaches are given in more detail in Li & Liu's book "Meshfree Particle Methods".

2.1.1 Drawbacks of SPH

One of the problems with SPH is that it does not reproduce simple behaviour particularly well. The **consistency conditions of order k** for a function w_h require that

$$\int_{\Omega} w_h(\mathbf{x} - \mathbf{y}) p_i(\mathbf{y}) d\mathbf{y} = p_i(\mathbf{x}), \quad i = 0, 1, 2, \dots, q. \quad (10)$$

where the set of polynomials $p_i(\mathbf{x}), i = 1, 2, \dots, q$ is a complete polynomial basis of order k . These conditions mean that the approximation function reproduces polynomial behaviour.

It can be shown that the consistency conditions are equivalent to requiring that

$$\int_{\Omega} w_h(\mathbf{x} - \mathbf{y}) p_m(\mathbf{x} - \mathbf{y}) d\mathbf{y} = p_m(\mathbf{0}), \quad m = 0, 1, 2, \dots, k. \quad (11)$$

The discretised forms then become

$$\sum_{i=1}^n w_h(\mathbf{x} - \mathbf{x}_i) p_m(\mathbf{x} - \mathbf{x}_i) \Delta V_i = p_m(\mathbf{0}), \quad m = 0, 1, 2, \dots, k. \quad (12)$$

It can be shown that SPH with a typical kernel function w_h will not meet these requirements for a general distribution of points.

SPH can have problems with shock waves and convective flows, but the problems can be avoided by introducing an artificial viscosity to damp out spurious oscillations (a similar approach is commonly used in FE simulation of high-speed impacts).

SPH suffers from tensile instability. Tensile instability means that if a region is in a tensile state and the positions of the points in the region are perturbed slightly, the perturbation can grow without bound.

As is the case for most meshless methods, it is not easy to enforce the essential boundary condition in SPH

2.2 Reproducing Kernel Particle Methods

Reproducing Kernel Particle methods (RKPM) are an attempt to use a similar approach to SPH, but to enforce consistency so that simple functions are reproduced by the method. The kernel function is written as a product of two functions: $w(\mathbf{x} - \mathbf{y})$ and $C(\mathbf{x}, \mathbf{y})$, a correction function. The approximation is then written as

$$u_h(\mathbf{x}) = \sum_{i=1}^n u_i w_h(\mathbf{x} - \mathbf{x}_i) C_h(\mathbf{x}, \mathbf{x}_i) \Delta V_i. \quad (13)$$

The correction function C_h is defined to have the form

$$C_h(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^q p_i(\mathbf{x} - \mathbf{y}) b_i(\mathbf{x}, h), \quad (14)$$

where the $b_i(x, h)$, $i = 1, 2, \dots, q$ are a set of unknown functions to be determined and the p_i , $i = 1, 2, \dots, q$ form a complete polynomial basis for the space. This form can also be written $C_h(x, y) = \mathbf{p}^T(\mathbf{x} - \mathbf{y}) \mathbf{b}(\mathbf{x}, h)$, where $\mathbf{p}$ is a column vector.

Applying the consistency conditions to this formulation gives

$$\sum_{i=1}^n \mathbf{p}^T(\mathbf{x} - \mathbf{x}_i) \mathbf{b}(\mathbf{x}, h) w_h(\mathbf{x} - \mathbf{x}_i) p_m(\mathbf{x} - \mathbf{x}_i) \Delta V_i = \delta_{m0}, m = 0, 1, 2, \dots, q, \quad (15)$$

or, rearranging slightly,

$$\sum_{j=0}^r b_j(\mathbf{x}, h) \sum_{i=1}^n (\mathbf{x} - \mathbf{x}_i)^j w_h(\mathbf{x} - \mathbf{x}_i) p_m(\mathbf{x} - \mathbf{x}_i) \Delta V_i = \delta_{m0}, m = 0, 1, 2, \dots, q, \quad (16)$$

or, in matrix form,

$$M \mathbf{b} = \mathbf{p}(0), \quad (17)$$

where M is the moment equation given by

$$M_{mj} = \sum_{i=1}^n p_j(\mathbf{x} - \mathbf{x}_i) w_h(\mathbf{x} - \mathbf{x}_i) p_m(\mathbf{x} - \mathbf{x}_i) \Delta V_i \quad (18)$$

or

$$M = \sum_{i=1}^n \mathbf{p}(\mathbf{x} - \mathbf{x}_i) \mathbf{p}^T(\mathbf{x} - \mathbf{x}_i) w_h(\mathbf{x} - \mathbf{x}_i) \Delta V_i. \quad (19)$$

Then the unknown functions can be written as

$$\mathbf{b} = M^{-1} \mathbf{p}(0) \quad (20)$$

so that the approximation function can be written

$$u_h(\mathbf{x}) = \sum_{i=1}^n u_i \mathbf{p}^T(\mathbf{x} - \mathbf{x}_i) M^{-1} \mathbf{p}(0) w_h(\mathbf{x} - \mathbf{x}_i) \Delta V_i, \quad (21)$$

or, defining

$$N_i(\mathbf{x}) = \mathbf{p}^T(\mathbf{x} - \mathbf{x}_i) M^{-1} \mathbf{p}(0) w_h(\mathbf{x} - \mathbf{x}_i) \Delta V_i, \quad (22)$$

the approximation function can be written

$$u_h(\mathbf{x}) = \sum_{i=1}^n N_i(\mathbf{x}) u_i. \quad (23)$$

2.2.1 Drawbacks of RKPM

RKPM solves many of the problems associated with SPH, but it still has some drawbacks. The essential boundary condition is still difficult to implement. RKPM is applied to the strong form of the equation and hence requires calculation of higher-order derivatives of the basis functions than those based on the weak form.

The method requires a connectivity array so that it is known which particles are interacting via their domains of influence. This array is computationally expensive to calculate. As points move, the coefficients in the matrix of discretised equations vary, and so a (computationally expensive) matrix inversion is required at every time step.

2.3 Element-free Galerkin methods

Element-free Galerkin (EFG) methods are not usually derived from an attempt to approximate the delta function. Instead the approximation is defined by a form similar to that of the correction function C_h in the RKPM. A set of points $\mathbf{x}_i, i = 1, 2, \dots, n$ is defined, and a local approximation is defined around each point. The local approximation $u_{h,L}$ is written

$$u_{h,L}(\mathbf{x}, \mathbf{x}_i) = \sum_{j=1}^q p_j(\mathbf{x}_i) a_j(\mathbf{x}, h) \quad (24)$$

where $p_j, j = 1, 2, \dots, q$ form a complete polynomial basis of order k and the a_j are a set of unknown functions to be determined. This form can be written $\mathbf{p}^T(\mathbf{x}_i) \mathbf{a}(\mathbf{x})$. The overall approximation to u is written

$$u_h(\mathbf{x}) = \sum_{j=1}^q p_j(\mathbf{x}) a_j(\mathbf{x}, h). \quad (25)$$

Note that whilst this expression is only dependent on $\mathbf{x}$ explicitly, there is a dependence on the points $\mathbf{x}_i$ because the unknown functions $a_j(\mathbf{x}, h)$ will be dependent on the $\mathbf{x}_i$.

The approximation minimises the weighted least squares difference between the value of each of the local approximations at $\mathbf{x}$ and a set of nodal parameters $u_i, i = 1, 2, \dots, n$ which are determined when the governing equation is solved. The weighted sum to be minimised is written

$$J(\mathbf{x}) = \sum_{i=1}^n w(\mathbf{x} - \mathbf{x}_i) [u_i - u_{h,L}(\mathbf{x}, \mathbf{x}_i)]^2. \quad (26)$$

The weighting function w is a smooth function with compact support and is often one of (5), (6), or (7). The i^{th} nodal parameter is effectively the value of the local approximation associated with the i^{th} point, evaluated at the point $\mathbf{x}$. This parameter is not the same as the global approximation to u at the point $\mathbf{x}$, which is how the nodal parameters in finite element analysis can be interpreted.

Substituting the chosen form of $u_{h,L}$ into the expression to be minimised gives

$$J(\mathbf{x}) = \sum_{i=1}^n w(\mathbf{x} - \mathbf{x}_i) [u_i - \mathbf{p}^T(\mathbf{x}_i) \mathbf{a}(\mathbf{x})]^2, \quad (27)$$

Defining a pair of matrices P and W and a vector $\mathbf{u}$ such that

$$P_{ij} = p_j(\mathbf{x}_i), \quad (28)$$

$$W_{ij} = \delta_{ij} w(\mathbf{x} - \mathbf{x}_i) \quad (29)$$

$$\mathbf{u}^T = (u_1, u_2, \dots, u_n), \quad (30)$$

the expression to be minimised can be written

$$J(\mathbf{x}) = (\mathbf{P} \mathbf{a}(\mathbf{x}) - \mathbf{u})^T \mathbf{W}(\mathbf{x}) (\mathbf{P} \mathbf{a}(\mathbf{x}) - \mathbf{u}), \quad (31)$$

or, in subscript notation,

$$J(\mathbf{x}) = (P_{ij} a_j(\mathbf{x}) - u_i) W(\mathbf{x})_{ik} (P_{kl} a_l(\mathbf{x}) - u_k), \quad (32)$$

The aim of the minimisation is to find the unknown functions a_j , and so a set of equations are derived by setting $\partial J / \partial a_j = 0$, $j = 1, 2, \dots, q$:

$$2P_{ij} W(\mathbf{x})_{ik} (P_{kl} a_l(\mathbf{x}) - u_k) = 0, \quad j = 1, 2, \dots, q, \quad (33)$$

and so, expanding the brackets and switching to matrix-vector notation,

$$\mathbf{P}^T \mathbf{W}(\mathbf{x}) \mathbf{P} \mathbf{a}(\mathbf{x}) - \mathbf{P}^T \mathbf{W}(\mathbf{x}) \mathbf{u} = 0. \quad (34)$$

The solution to these equations can then be written as

$$\mathbf{a}(\mathbf{x}) = (P^T W(\mathbf{x}) P)^{-1} P^T W(\mathbf{x}) \mathbf{u}, \quad (35)$$

and so by substituting (35) into (25) the approximation function can be written as

$$u_h(\mathbf{x}) = \mathbf{p}^T(\mathbf{x}) (P^T W(\mathbf{x}) P)^{-1} P^T W(\mathbf{x}) \mathbf{u}. \quad (36)$$

Defining

$$N_i(\mathbf{x}) = \mathbf{p}^T(\mathbf{x}) (P^T W(\mathbf{x}) P)^{-1} (P^T W(\mathbf{x}))_{ki} \quad (37)$$

means that the approximation can be written

$$u_h(\mathbf{x}) = \sum_{i=1}^n N_i(\mathbf{x}) u_i. \quad (38)$$

The approximation is then applied to the weak form of the equation, so a set of test functions is required. Typically the test functions are chosen to be the shape functions N_i , but this is not a requirement for the methods to be successful.

2.3.1 Drawbacks of EFG methods

As with SPH and RKPM, it is not simple to apply the essential boundary condition to EFG methods.

EFG methods use the weak form of the equation, which requires evaluation of integrals to generate the discretised form. In general the integrals cannot be evaluated analytically, so numerical quadrature is necessary. Since the shape functions and test functions for EFG are more complicated than the polynomials used in FE, it is possible that a large number of quadrature points will be required to evaluate the integrals accurately.

2.4 Material point methods

Material point methods (MPM) are an alternative way of addressing the problems associated with large deformations. They evolved from a set of methods known as “particle-in-cell” methods that were developed for fluid mechanics problems. The methods are similar in some ways to an Eulerian approach in as much as they simulate the flow of material through a fixed grid.

The methods require a set of points and a grid. The grid entirely fills the computational domain and is usually designed for computational convenience. Typically the grid will

remain fixed during calculations, but a new grid can be generated mid-simulation if that approach is beneficial. The material points move during the simulation. Each point has a set of values associated with it that it carries as it moves. For a transient stress analysis, these values would include mass, density, velocity, acceleration, stress, and strain.

At each time step, the mass and momentum values at each material point are mapped to the nodes of the grid cell containing that point, generally using the standard FE basis functions for the given cell shape, so that

$$q_i = \sum_{n=1}^{N_p} Q_p N_i(\mathbf{x}_n), \quad (39)$$

where q_i is the value of a quantity at the i^{th} grid node, Q_n is the value of a quantity at the n^{th} material point $\mathbf{x}_n$, N_p is the number of material points, and N_i is the basis function associated with the i^{th} grid node. The forces on each grid node are calculated from the stresses and the boundary conditions, and are used to update the momentum values at the grid nodes. The forces and the new momentum values are then used to calculate new accelerations and velocities for the material points using the same shape functions as (39). The velocity is used to calculate an updated position and displacement, from which strains and stresses are calculated.

2.4.1 Drawbacks of MPM

The method requires more storage space than some methods as it requires a grid as well as a set of points, and the points typically have more information associated with them than in some methods. The original form of the method can develop spurious oscillations when particles cross cell boundaries, although this drawback can be addressed by using a better mapping method than that outlined above.

2.5 Application of the essential boundary condition

As has been mentioned above repeatedly, the majority of meshless methods struggle to apply the essential boundary condition. Ways of handling this problem are addressed by some authors [4, 11]. One possibility is to couple the meshless method to a finite element method, which is ideally suited to applying the essential boundary condition. Some methods for carrying out this coupling are described in section 3.

One commonly-used method is to introduce Lagrange multipliers, provided that the problem to be solved is being expressed in the weak form. A set of terms of the form $\lambda_i(u_i - u_B)$ is added to the discretised weak form, where λ_i is a Lagrangian multiplier to be determined, u_i is the unknown quantity at a point on the boundary, and u_B is the

imposed boundary value. One term is included per point to be constrained. The terms will vanish when the boundary conditions are satisfied. The multipliers are extra variables that must be solved for, so that problem size increases and the matrix of equations changes. The method is easy to implement but can be computationally expensive.

Another approach is to add a penalty function $\alpha \int (u_i - u_B) dx$ to the weak formulation, where α is a (fixed, known) large number. This approach can lead to numerical problems since the resulting matrix of equations can become ill-conditioned.

Transformation methods reformulate the problem, either entirely or partially, by attempting to determine directly the true value of the unknown quantity at the set of points rather than determining a set of parameters associated with the points that allow the true value to be calculated. For instance, if the discretised form of the equations that determines the parameters is $A\hat{\mathbf{u}} = \mathbf{b}$ and the true solution is determined from $\mathbf{u} = D\hat{\mathbf{u}}$, then the true solution can be determined directly by solving $AD^{-1}\mathbf{u} = \mathbf{b}$. This approach can be applied to the entire set of points or only to those nodes that lie on the boundary. The main drawback of the method is computational expense since it requires an extra matrix inversion step, and the matrix to be inverted is generally full and not symmetric, so its inversion is non-trivial.

A simple and direct approach is to impose the conditions within the matrix of equations. If the boundary condition is to be applied at the point $\mathbf{x}_j$, then the equation

$$u(\mathbf{x}_j) = \sum_{i=1}^n u_i N_i(\mathbf{x}_j) = u_B \quad (40)$$

can be included in the full matrix. This method enforces the boundary condition at nodal points, but not between the points. The method also breaks an important condition on the test functions. Test functions must have a value of zero where essential boundary conditions are applied. The removal of this constraint can lead to a reduction in the convergence order.

2.6 Software

An internet search has been carried out for software implementations of meshless methods, but very little has been found. Some commercial packages include meshless methods in their more recent releases: LS-DYNA, commonly used to simulate impacts and other dynamic events, includes SPH and EFG capabilities.

A freeware package “MFree2D” is available from <http://www.nus.edu.sg/ACES/software/meshless2D/webfiles/webpageMFree.htm>. The package carries out two-dimensional elastostatic simulations. It is not immediately clear from the website which of the methods described above are used in the software.

A Matlab implementation of the EFG method applied to a bending beam in one and two dimensions is available from <http://www.duke.edu/~jdolbow/EFG/programs.html>

A book, “Meshfree Approximation Methods With MATLAB” by G. E. Fasshauer, with a CD of Matlab implementations of meshless methods is available. The book appears to focus mostly on a radial basis function approach to meshless methods.

Various libraries for calculation of EFG shape functions are available from <http://hogwarts.ucsd.edu/~pkrysl/software.html>.

An implementation of the decomposed EFG method for solution of elastic wave equations can be found at <http://software.seg.org/2009/0002/index.html>

3 Coupling between meshless methods and FE

It has been noted throughout this document that meshless methods struggle to impose the essential boundary condition. FE methods have no such problem, because they allow for direct control of variable values at a point. It has also been stated that meshless methods may be computationally expensive due to the need to recalculate the matrix of equations regularly, and in some cases due to the potentially computationally expensive numerical quadrature. FE methods are generally less computationally expensive as the simple shape functions require fewer quadrature points for accurate calculation of integrals.

An area of increasing interest is the linking of meshless and FE methods. This would allow meshless methods to be used in the regions of a domain where they would be of most benefit, e.g. in the direct vicinity of a crack, and FE methods to be used elsewhere, particularly in the regions where the essential boundary conditions are important. It would also allow commercial FE software manufacturers to incorporate meshless methods into their software with the minimum amount of redesign.

Various methods for coupling meshless and finite element methods have been proposed. Many of the methods have been compared by Rabczuk et al [23] by application to several static and dynamic examples, including crack propagation. This section describes some of the coupling methods investigated by Rabczuk et al [23] and others [4, 22, 11]. Most of the methods as defined here are formulated for stress-strain problems, but the methods could be applied to other problems with appropriate minor reformulations.

3.1 Master-slave coupling

Master-slave coupling techniques treat the particles as slave nodes whose movement is determined by the movement of the FE nodes. In the simplest such technique, the domain is divided into a meshless region and an FE region, and the nodes on the boundary between the FE region and the meshless region are treated as meshless particles and FE nodes simultaneously. The force $\mathbf{F}_P$ acting on the particle, mass m_P , due to interaction with other particles is calculated, as is the force $\mathbf{F}_N$ acting on the node, mass m_N , due to interaction with other nodes, and the overall acceleration is taken to be

$$\mathbf{a} = \frac{\mathbf{F}_P + \mathbf{F}_N}{m_P + m_N}. \quad (41)$$

This formulation is not suitable for general distributions of particles and so may become inadequate as the interface deforms. In addition, the restriction of particle distribution may mean that the method does not reproduce simple behaviour correctly.

A more general form of master-slave coupling that allows for an arbitrary distribution of meshless particles is to couple the particles nearest to the boundary (slave particles) to the surface of the elements on the boundary. The velocities of the slave particles are calculated in terms of the nodal velocities of the FE mesh, and the forces acting on the slave particles are added to the nodal forces of the boundary elements.

According to Rabczuk et al [23], the master-slave coupling method is the most straightforward to implement, but their test results suggest that the method produces the least accurate results of the methods tested for linear elasto-static problems. When the methods are applied to more challenging problems such as crack propagation, the difference in accuracy is reduced. The method may suffer from error accumulation at long calculation times for dynamic problems.

3.2 Coupling via ramp functions

Coupling via ramp functions splits the domain into three regions: the meshless region, where the approximate solution is $u_P(\mathbf{x})$, the FE region, where the approximate solution is $u_{FE}(\mathbf{x})$, and an interface region. The interface region has a set of elements defined on it so that the FE shape functions are well-defined within that region. In the interface region, the solution is given by

$$u_{IF}(\mathbf{x}) = R(\mathbf{x})u_P(\mathbf{x}) + (1 - R(\mathbf{x}))u_{FE}(\mathbf{x}) \quad (42)$$

where R is a ramp function that is zero on the boundary between the meshless region and the interface region, and uniformly 1 on the boundary between the FE region and the interface region. The sum of all the shape functions of the elements defined in the interface

region is zero on one boundary and takes the value 1 everywhere on the other, and so the ramp function can be written as

$$R(\mathbf{x}) = f\left(\sum_{I \in I_{IF}} N_I(\mathbf{x})\right), \quad (43)$$

where I_{IF} is the set of element labels of the elements in the interface region and f is such that $f(0) = 0$ and $f(1) = 1$. The simplest form of f is to let R be equal to the sum of FE shape functions, but Rabczuk et al [23] suggest that

$$f(r(\mathbf{x})) = 3r^2 - 2r^3 \quad (44)$$

where

$$r(\mathbf{x}) = \sum_{I \in I_{IF}} N_I(\mathbf{x}), \quad (45)$$

is an appropriate choice, possibly because it has zero gradients at both ends of the domain. The test functions for the weak form of the equation should have the same shape as u_{IF} within the interface region.

The ramp functions can have a complicated shape (described by Belytschko et al [4] as “baroque”), which may lead to increased computational expense if the quadrature for the functions becomes expensive. Construction of the approximation within the interface domain is likely to be challenging for EFG methods.

3.3 Coupling with reproducing conditions

It has been pointed out earlier in this document that consistency is an important feature if a method is to reproduce simple polynomial behaviour correctly. A coupling approach has been developed that allows a consistent method of any order to be constructed. As with the ramp function method, the domain is split into three parts (meshless, FE, and interface). The main difference between this method and the ramp function method is that FE shape functions are defined only for nodes within the FE region (internal and border), but not the interface region. Meshless shape functions are defined for all nodes not in the FE region (i.e. all those in the meshless and interface regions apart from those on the border between the FE and interface regions).

Writing $u_P(\mathbf{x})$ for the solution in the meshless region containing nodes I^P , $u_{FE}(\mathbf{x})$ for the solution in the FE region containing nodes I^{FE} , and $u_{IF}(\mathbf{x})$ for the solution in the interface region, the aim is to make $u_{IF}(\mathbf{x})$ consistent without altering the FE shape functions. The full derivation of the equations is given in Fries & Matthies [11]. The main steps are expanding the solution as a Taylor series and writing the meshless shape functions in full, imposing the consistency conditions by considering the Taylor series terms, and solving for unknown quantities.

For a meshless approximation of the form

$$u_P(\mathbf{x}) = \sum_{i=1}^N \mathbf{p}^T(\mathbf{x}_i - \mathbf{x}) \mathbf{a}(\mathbf{x}) w(\mathbf{x}_i - \mathbf{x}) u_i, \quad (46)$$

the resulting formulation is

$$\begin{aligned} u_{IF}(\mathbf{x}) &= \sum_{i \in I^P} \left(\left(\mathbf{p}^T(\mathbf{x}) - \sum_{i \in I^{FE}} N_i(\mathbf{x}) \mathbf{p}^T(\mathbf{x}_i) \right) M^{-1}(\mathbf{x}) \mathbf{p}(\mathbf{x}_i) w(\mathbf{x}_i - \mathbf{x}) u_i \right) \\ &+ \sum_{i \in I^{FE}} N_i(\mathbf{x}) u_i \end{aligned} \quad (47)$$

where N_i are the FE shape functions and $M(\mathbf{x}) = \sum_{i \in I^P} w(\mathbf{x}_i - \mathbf{x}) \mathbf{p}(\mathbf{x}_i) \mathbf{p}^T(\mathbf{x}_i)$ is the moment matrix.

This approach results in unusually-shaped basis functions in the interface region, which may make the method computationally expensive.

3.4 Bridging domain methods

The bridging domain method as described in Rabczuk et al [23] is based on the same subdivision of the domain as the coupling with reproducing conditions outlined above. In the derivation given in Rabczuk et al [23], the method is restricted to mesh and particle arrangements where the interface region is a rectangle, but it may be possible to generalise the concept to other geometries. The method is derived specifically for deformation problems, but a change of the quantity being minimised during the derivation would generate a version of the method suitable for other problems.

The method is derived by defining a scaling parameter $\alpha(\mathbf{x})$ within the interface region as $\alpha(\mathbf{x}) = l(\mathbf{x})/l_0$ where $l(\mathbf{x})$ is a measure of how far into the interface domain $\mathbf{x}$ is, so that if $\mathbf{x}$ lies on the boundary between the interface region and the FE region, $l(\mathbf{x}) = 0$, and if $\mathbf{x}$ lies on the boundary between the interface region and the particle region, $l(\mathbf{x}) = l_0$. Hence α takes a value of 1 on one boundary, 0 on the other, and varies linearly in between. A function β can then be defined as

$$\beta(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \in \Omega^{FE}, \\ 1 - \alpha & \mathbf{x} \in \Omega^{IF}, \\ 0 & \mathbf{x} \in \Omega^P, \end{cases} \quad (48)$$

where Ω^{FE} , Ω^{IF} and Ω^P are the FE, interface, and particle domains respectively.

The internal energy of a body undergoing deformation in the absence of heat transmission is written as

$$W^{int} = \beta W_{FE}^{int} + (1 - \beta) W_P^{int} \quad (49)$$

where W_{FE}^{int} is the internal energy of the FE domain, including the energy calculated from the FE approximations within the interface domain, and W_P^{int} is the internal energy of the particle domain, including the energy calculated from the meshless approximations within the interface domain.

Other similar expressions are written for the external energy and the boundary conditions. Minimisation of the sum of the energy terms, including the boundary conditions via Lagrange multipliers, with respect to the unknown parameters in the FE and particle methods is used to derive a full set of discrete fully coupled equations that are dependent on α . These equations can be solved in the same way as meshless or FE discretised equations.

This coupling technique should not be any more computationally expensive than the individual methods, but will require careful formulation of the coupled problem.

3.5 Coupling with Lagrange multipliers

Coupling with Lagrange multipliers is derived from the minimisation of an extended energy potential. A similar approach is often used for implementation of derivative boundary conditions within an FE formulation. The FE and meshless methods are used on two disjoint regions with no overlap and a common boundary Γ . The extended energy potential is written as

$$W = W^{int} - W^{ext} + \lambda g, \quad (50)$$

where W^{int} is the internal energy (e.g. internal strain energy) and W^{ext} is the external energy (e.g. due to body loads or applied stresses). The function g is defined along Γ as the difference between the FE and meshless solutions,

$$g(\mathbf{x}) = u_{FE}(\mathbf{x}) - u_P(\mathbf{x}). \quad (51)$$

The Lagrange multiplier λ is an unknown quantity that is typically parameterised using the FE shape functions and evaluated at the particle positions along Γ . Minimisation of the extended potential via test functions leads to a full set of coupled discrete equations that include parameters associated with λ as unknowns.

This coupling technique is computationally expensive as it requires extra variables to take the coupling into account. It may be complicated to set up a fully coupled formulation.

3.6 Criteria for judging coupling methods

The main requirement of a coupling method is that no spurious loads or constraints must be introduced by the method. The method must create a solution that is continuous in the

variable of interest, that still obeys the governing pdes, and that does not introduce forces, fluxes, or other phenomena with no physical basis.

Ideally the coupling between two methods would be of the same order of accuracy (in terms of reproducing conditions or truncation/discretisation error) as at least the less accurate method. Similarly if the coupled methods both have smooth or continuous derivatives up to some order, then ideally the coupling method should preserve that smoothness or continuity.

Most of the methods listed above create a coupling by defining a new functional form for the variable of interest within the coupling domain, and applying the governing pde to derive a new set of discretised equations. This approach means that the solution is sure to obey the governing pde. Many of the pdes used in physics models are derived from conservation laws that ensure that no spurious forces or similar effects are introduced, and so the coupled formulations should avoid spurious loading problems.

A methodology for testing coupled problems has been derived as part of the work in the Hybrid Methods project. The methodology defines a set of test problems that can be used to assess the “matchedness” of coupled methods. The methodology was designed for problems where mismatches in the definition of physical properties are likely, and particularly for problems that couple molecular or atomic methods to continuum methods. The methodology is also applicable to coupling of different continuum methods and would be a good way of testing the validity of the coupling methods listed above.

4 Recommendations for further work

Of the methods described above, EFG methods offer a number of advantages, particularly when the problems of coupling with FE methods are considered. The methods are formulated using the weak form of the equation, as are FE methods, meaning that coupling methods that impose reproducing conditions are likely to be more straightforward using EFG than other methods. A piece of freeware Matlab code and explanatory paper is available that solves a plane stress problem which could be adapted to solve the problems used in the “matchedness” testing methodology. The method requires less determination of derivatives. The main drawback is the computational expense of the numerical quadrature.

All of the coupling methods listed in section 3 could be used to couple an EFG implementation to an FE implementation. It is likely that the following list places the methods in increasing order of difficulty of implementation:

1. Master-slave coupling.
2. Coupling with Lagrange multipliers.

3. Coupling with ramp functions.
4. Coupling with reproducing conditions.
5. Bridging domain methods.

The methods should be tested using the “matchedness” testing methodology in this order.

5 Literature review

This section aims to survey existing literature on meshless methods, including key review papers and papers applying methods to real problems. Some of the literature was obtained from internet searches and has, in some cases, since been removed, but in general the authors are still active in the field and have produced published papers with similar titles to the work listed.

5.1 Review papers

- “Meshfree and particle methods and their applications”, S. Li, W. K. Liu [14]
Review of meshless methods. Section on SPH, including general summary, list of problems with the method, description of techniques used to overcome those problems (e.g. an inability to simulate rigid body motions led to development of the RKPM), and a list of software codes (mostly parallel). Section on mesh-free Galerkin methods, including history and general summary, proof of consistency and convergence, h and p enrichment of the methods, problems with the method (imposition of essential boundary conditions, quadrature issues) and a lengthy list of applications. Section on ab initio methods and molecular dynamics (since these are particle methods too), including details of ab initio methods, density functional theory, ab initio molecular dynamics, classical molecular dynamics, and applications. Section on other particle methods: vortex methods for CFD, particle-in-cell methods, lattice Boltzmann method, natural element methods (based on Voronoi and Delaunay tessellations). 397 references.
- “Meshfree particle methods”, S. Li, W. K. Liu [15].
Book addressing derivation and implementation of meshfree particle methods. Relevant chapter titles: Smoothed Particle Hydrodynamics, Meshfree Galerkin Methods, Approximation Theory of Meshfree Interpolants, Applications, Reproducing Kernel Element Method, Molecular dynamics and Multiscale Methods, Immersed meshfree/Finite element Method and Applications, Other Meshfree Methods. Fortran listing of RKPM applied to the Helmholtz equation on a 1D bar. 502 pages, 475 references.

- “Review of development of the smooth particle hydrodynamics (SPH) Method”, R. Vignjevic [29].

Explains SPH formulation, derives discrete forms of conservation laws (mass, momentum, energy) for the formulation, discusses choice of kernel function, smoothing distance and neighbour search methods, identifies shortcomings in the method and describes methods of addressing the shortcomings. 62 references.

- “Meshless methods and their computer implementation aspects” N. V. Phu [22].

Source unknown, but probably contains much of the material in “Meshless methods: a review and computer implementation aspects”, N. V. Phu, T. Rabczuk, S. Bordas and M. Duflot, *Mathematics and Computers in Simulation* Volume 79 , Issue 3, 763-813, 2008.

Broad review of meshless methods and the difficulties of implementation. Introduces basic concepts of moving least squares interpolants and partition of unity methods and develops the weak form as the minimisation of weighted residuals. Explains specific methods (EFG, SPH, RKPM) and discusses problems associated with meshless methods (essential boundary conditions, integration, discontinuities). Discusses two methods for coupling with FE in detail. Extensive section on how to write an EFG code, including example Matlab code and advice on improving computational efficiency. Results of four example problems from elastostatics are shown. Full Matlab listing for a 2D EFG code for a Timoshenko beam is given. 30 references.

- “Meshfree methods”, G. E. Fasshauer [10]. Source unknown, but the author has extensive publications in meshless methods and radial basis functions, including a book “Meshfree approximation methods with Matlab”.

Review of meshless interpolation methods that use radial basis functions. Introduces radial basis functions and derives many properties of that class of function. Applies such functions to least squares and moving least squares approximation. Discusses issues around practical implementation of the methods. Application to solution of pdes only introduced late on with most of the early sections focussing on functional analysis. 215 references.

- “Classification and overview of meshfree methods”, T.-P. Fries, H.-G. Matthies [11].

Classifies methods according to their choice of construction of a partition of unity, choice of approximation basis, and choice of test function. Explains each of these concepts, and some common choices for each, in detail. Gives explanations of a range of methods. Discusses problems associated with the methods and offers solutions of those problems, including four methods for coupling mesh-based and meshfree methods and a discussion of parallelisation. 133 references.

- “Meshless methods: an overview and recent developments” T. Belytschko, Y. Krongauz, D. Organ, M. Fleming, P. Krysl [4]

Possible that online document and journal article are not the same since online document has 79 pages to the article's 43, but authors and title are identical. Summary here is of online version.

Discusses construction and consistency of a variety of methods. Gives details of how to handle problems involving discontinuities in functions and in derivatives. Outlines numerical implementation procedure. Discusses coupling to FE methods and computational cost. Investigates convergence through a convergence study and shows a range of other examples. 63 references.

- “Coupling of meshfree methods with finite elements: basic concepts and test results”, T.Rabczuk, S.P.Xiao, M.Sauer [23].

Reviews coupling methods for stress analysis applications. Defines common shape functions in use in FE and meshless methods. Describes a range of coupling methods in detail. Applies the methods to four examples (two static and two dynamic), including a crack propagation problem. Results suggest that the best choice of coupling method is problem-dependent. 51 references.

5.2 Applications

- “Material point method calculations with explicit cracks, fracture parameters, and crack propagation”, J. A. Nairn and Y. J. Guo [20].

Describes a method for calculating crack propagation that allows for inclusion of explicit cracks and arbitrary propagation direction, based on a material point method. Demonstrates calculation of J integral and stress intensity factors as well as displacements and crack behaviour. Outlines method without giving explicit details. Other reference by same authors (Nairn, J. A., Material Point Method Calculations with Explicit Cracks, Computer Modeling in Engineering & Sciences, 4, 649-664, 2003) is likely to supply full details but has not been obtained as yet.

- “Three-dimensional dynamic fracture analysis using the material point method”, Y. J. Guo and J. A. Nairn [12].

Gives full details of a three-dimensional version of the method outlined above. Includes calculations of J-integrals and stress intensity factors, and considers contact between the crack surfaces. Compares results for three different problems to results obtained using FE and finite difference methods.

- “An element-free Galerkin method for three-dimensional fracture mechanics”, N. Sukumar, B. Moran, T. Black, T. Belytschko [28].

Describes a coupled EFG-FE method for solution of crack problems. An enriched EFG formulation is used in the region of the crack tip in order to approximate the singularity accurately. The diffraction method is used to provide smooth shape functions for points whose domain of influence includes the crack surface. The

coupling between methods is implemented using blending functions that preserve the continuity of displacement values. The method is applied to the calculation of displacements and stress intensity factors on a single edge-crack tension specimen and a penny-shaped crack in an infinite body.

- “Fracture and crack growth by element free Galerkin methods”, T. Belytschko, L. Gu and Y. Y. Lu [3].

Comparatively early paper describing the derivation of the EFG method and its application to fracture problems. Illustrated by application to three fracture problems in 2D.

- “Integrating mesh and meshfree methods for physics-based fracture and debris cloud simulation”, N. Zhang, X. Zhou, D. Sha, X. Yuan, K. Tamma, and B. Chen [31].

Uses FE and an EOS material model to simulate deformation and failure of a vase being dropped. Once a given element or group of elements have failed, it is subsequently treated as a particle of equivalent mass, and its trajectory is modelled using Newton’s second law. Interactions between particles are modelled using a potential energy approach. Application of interest is graphical representation of a fragmenting object and so validation is restricted to “looks right”.

- “Semi-adaptive coupling technique for the prediction of impact damage”, L. Aktay, A. F. Johnson, and B.-H. Kröplin [1].

Outlines a technique that couples FE and SPH to simulate failure of sandwich panels under impact. When elements reach a failure criterion they are replaced with an SPH formulation that includes contact so that the damaged material continues to resist the impact. Model results are compared to plain FE results (with complete removal of damaged material) and to experiment, and it is shown that the coupled technique performs better than FE alone. It is not clear why the title calls this a “semi-adaptive” coupling.

- “Application of the Fictitious Crack Model to meshless crack growth simulations”, T. Most & C. Bucher [19].

Gives full details of an MLS meshless method coupled with FE to study damage in concrete. The fictitious crack model allows force transmission across the crack surface for micro cracks but not macro cracks. Micro cracks occur when a maximum principal tensile stress is exceeded. Macro cracks are created when a micro crack width exceeds a specified value. Crack growth is controlled by strain at the crack tip. The problem is initially formulated as an FE problem, and when a micro crack occurs within an element, the element is converted into an MLS formulation. Results of a simulation of a reinforced concrete bar under load are compared to experiment.

- “A comparison of meshless and finite element approaches to ductile damage in forming processes”, J. M. A. Cesar de Sa and C. Zheng [7].

Applies the RKPM to problems of ductile damage in forming processes. Uses a viscoplastic material formulation and a local damage model based on assumptions about distribution of microvoids. Several non-local damage models were also implemented using both formulations for comparison. For a simple tensile test, FE and RKPM gave similar results. Conclusions imply that there was no benefit in using RKPM over FE. I suspect this is due to choice of problem since FE seemed to have no trouble solving the problem and the point is to use RKPM when FE doesn't work.

- “Meshless method for modelling large deformation with elastoplasticity”, J. Ma [18].

Develops a meshless integral method based on the regularized boundary integral equation. Applies the method to problems in two-dimensional linear elasticity and elastoplasticity with small or large deformations. Full explanation of all aspects is included, including pre- and post-processing routines. Application examples include infinite plate with a circular hole and various simple tests validated against analytical theory and FE models.

- “The material point method for simulation of thin membranes”, A. R. York II, D. Sulsky, and H. L. Schreyer [13].

Extension of the MPM to handle membranes (i.e. structures significantly thinner in one dimension than the others that are resistant to in-plane loading but not bending: note that this definition includes strings as well as planar objects). Contact is included in the formulation. Various examples are shown and results are compared to theory and to FE results.

- “Application of meshless EFG method in fluid flow problems”, I. V. Singh [26].

Applied EFG to fluid flow. Derives discretised equations in full for flow down a rectangular duct. Compares results using various weight functions to FE results and the analytical solution for the same problem.

- “High-order fundamental and general solutions of convection-diffusion equation and their applications with boundary particle method”, W. Chen [6].

Uses high-order general and fundamental solutions to the modified Helmholtz equation to derive a boundary particle method for solution of convection-diffusion equations with constant coefficients. Method appears to suffer from ill-conditioning.

- “Radial point collocation method (RPCM) for solving convection-diffusion problems” X. Liu [17].

Uses a meshless method to solve transient convection-diffusion problems. Method uses radial basis functions and point collocation. Multiquadric basis functions are used, which are of the form $(\sqrt{r_i^2 + c_i^2})^Q$, where Q is a constant, r_i is the distance from the point of interest to a given collocation point, and c_i is some shape parameter associated with that collocation point. Implicit and Crank-Nicolson schemes are used for time integration. The technique is demonstrated on the a set of

examples with known solutions. The method requires a fictitious point approach for derivative boundary conditions since it is based on the strong form of the equation.

- “Spatial coupling in aeroelasticity by meshless kernel-based methods”, H. Wendland [30].

Uses meshless method to simulate interaction between elastic structure and fluid loading. Treated as a three field problem (fluid, structure and interface), fluid and structural problems are solved using finite volumes and finite elements respectively, and the coupling problem is solved using meshless methods. Radial basis functions are used to interpolate displacements from the structural model onto the mesh for the fluid model. Forces from the fluid model are transferred to the structural model via a transformation that conserves total force and work done by the forces.

- “Meshless difference methods with adaptation and high resolution”, Q. Lin, J. G. Rokne [16].

Applies meshless finite difference methods to problems with moving interfaces and moving boundaries. Allows for addition or deletion of nodes during calculation, hence adaptive, depending on the size of an error estimator. Technique is illustrated by application to three fluid-flow examples but results are poorly explained. Despite claiming to be meshless, has a fairly rigorous approach to connectivity via node “stencils” and the method appears to be a generalised FD method rather than a meshless one.

- “A hybrid meshless-collocation/spectral-element method for elliptic partial differential equations”, C. D. Blakely [5].

Couples a spectral element method with a reproducing kernel collocation method to solve elliptic equations, including a non-homogeneous Helmholtz problem. Has background information on coupling methods via the three-field technique. Includes full proof that the method creates a unique solution to the problem. Much of the functional analysis is challenging.

- “Meshfree simulation of deforming domains”, V. Shapiroy, I. Tsukanovz [25].

Describes the R-function method and its application to problems with moving domains, specifically heat transfer, within a piston-cylinder assembly. Approach writes solution of a given model with boundary conditions as the product of a function that satisfies the conditions on the boundary, constructed from R-functions, and a function to be approximated in the interior of the domain using some set of basis functions, the coefficients of which are determined during the solution procedure. The method is validated against a simple 1D problem. Difficulties of implementation are addressed.

- “Meshless methods and partition of unity finite elements”, N. Sukumar [27].

Explains and compares EFG, natural neighbour method and partition of unity FE. No examples.

- “Implementation of meshless FEM for engineering applications”, A. Seidl, and T. Schmidt [24]

Applies EFG, point interpolation methods and FE to electrical problems (not specified explicitly) in 2D. Explains structure of software class implementation used in detail.

- “Coupling projection domain decomposition method and meshless collocation method using radial basis functions in electromagnetics”, Y. Duan, S. J. Lai, T. Z. Huang [9]

Applies a meshless RBF collocation method to electrostatic and magnetostatic problems. Uses a projection domain decomposition to decrease calculation time. Validates against a problem with an analytical solution. No statistics are supplied for speed-up of performance improvement.

- “A hybrid meshless local PetrovGalerkin method for unbounded domains”, A. J. Deeks, C. E. Augarde [8].

Describes coupling a local meshless method to a meshless scaled boundary method to handle problems with unbounded domains. Reviews both methods in outline and describes the coupling method in full. Tested on two examples in 2D elastostatics that have analytical solutions. Method outperforms FE.

- “Element free Galerkin mesh-less method for fully coupled analysis of a consolidation process”, M. N. Oliaei and A. Pak [21].

Applies an EFG method to a coupled problem simulating transient behaviour of saturated soil. Summarises EFG, derives discretised form of equations for pore pressure and displacement of soil skeleton. Gives a step by step outline of software implementation. Compares results to analytical and FE results in 1 and 2D.

- “Local maximum-entropy approximation schemes: a seamless bridge between finite elements and meshfree methods” M. Arroyo and M. Ortiz [2].

Describes a new kind of approximation method that has Galerkin FE as a special example. Illustrated with examples. Seems to handle high-deformation problems well and doesn't have the problem with essential boundary conditions that meshless methods do. Proofs and derivations are quite challenging.

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