

NATIONAL PHYSICAL LABORATORY

Centre for Materials Measurement & Technology

DESIGN GUIDE FOR PREDICTING COMPOSITE PROPERTIES

PART I : Description of methodology

by

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SUMMARY

Part I of the design guide provides a brief description of the various models that may be used to predict reliably the effective thermo-elastic properties of unidirectional and laminated composites, for both undamaged and damaged states. The effective thermo-elastic properties can be predicted as a function of crack density using models of stress transfer that take account of the effects of thermal residual stresses. Energy methods are used to predict ply crack formation in multiple ply cross-ply laminates subject to multi-axial loading, and in general symmetric laminates subject to combined biaxial and shear in-plane loading.

N.B. Part II of the guide will address the validation of the methodology by comparison of predictions with both finite element analysis and experiment. Part III of the guide will indicate how the methodology, and associated computer software, may be used in the design of composite components.

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1. Introduction

The manufacture, testing and qualification of composite materials for engineering application is a very time consuming and expensive part of the development cycle for new composite components and structures. Reliable techniques for the prediction of the properties of composites based on the principles of mechanics, when combined with recent technical advances in the performance of PCs, can offer much scope for the reduction of some of these development costs. A description will be given of recent progress with the use of micro-mechanical models when developing such predictive techniques in an engineering environment.

There are three types of behaviour that need to be considered when designing composite components. First of all, for relatively low applied loadings the composite deforms as a linear elastic solid without the formation of any type of microstructural damage, i.e. all interfaces between plies in laminates, and between fibres and matrix in plies are perfectly bonded. Many models predicting the thermo-elastic constants of unidirectional and laminated composites have been developed for this type of behaviour. For larger applied loads the composite experiences the second type of behaviour where microstructural damage formation occurs leading to the progressive degradation of material properties, and to non-linear stress-strain behaviour. Engineering components usually exhibit stress concentrations arising from geometrical features such as holes, changes of section etc. Such concentrations cause localised microstructural damage formation and localised property degradation that leads to the macroscopic redistribution of stress within the component. This is a very important phenomenon that is largely responsible for composite components out-performing their expectations based on designs using laboratory coupon data for relatively simple loading modes. Following the progressive stable growth of microstructural damage, the third type of behaviour ensues where instability and catastrophic failure of the structure occurs. Important design objectives are to delay the occurrence microstructural damage, to extend the progressive micro-structural damage growth phase of deformation, and to delay the onset of catastrophic failure thus leading to engineering components that fail 'gracefully', i.e. with plenty of warning.

When using commercial computer codes to design composite components, the engineer has at his disposal a multitude of criteria to predict the failure of the structure. The calculations are usually based on the finite element method, and assume linear anisotropic behaviour up to the point of failure which is determined by the application of criteria that do not take account of the microstructural behaviour of the composite material. Such an approach does not account for progressive damage formation in components and the associated stress redistribution, and is not able to distinguish between different failure modes e.g. matrix cracking, delamination, fibre fracture, and therefore must lead to uncertainty when predicting performance. While it is very difficult to replace current failure criteria with more physically based models that can account in detail for differing failure modes, some good progress is being made and it is appropriate to indicate how such physical models now have a role to play in the design of composite components and structures.

This design guide is divided into three distinct parts where Part I briefly describes the characteristics of the various models developed. Part II (to be completed in the future) will address the very important issue of demonstrating model validity by comparing model predictions with both FEA analysis and experiment. Part III (to be developed on completion of Part II) will describe how the models may be used, together with associated software, in

the design process for composite components.

2. List of symbols

For isotropic fibres and matrix :

E_f	Young's modulus for fibre	E_m	Young's modulus for matrix
ν_f	Poisson's ratio for fibre	ν_m	Poisson's ratio for matrix
$\bar{\lambda}_f$	shear modulus for fibre	$\bar{\lambda}_m$	shear modulus for matrix
k_f	bulk modulus for fibre	k_m	bulk modulus for matrix
$\bar{\alpha}_f$	thermal expansion coefficient for fibre	$\bar{\alpha}_m$	thermal expansion coefficient for matrix
V_f	fibre volume fraction	V_m	matrix volume fraction ($V_m = 1 - V_f$)

For transverse isotropic unidirectional composites :

E_A	axial Young's modulus
E_T	transverse Young's modulus
ν_A	axial Poisson's ratio
ν_T	transverse Poisson's ratio
$\bar{\lambda}_A$	axial shear modulus
$\bar{\lambda}_T$	transverse shear modulus
k_T	transverse bulk modulus
$\bar{\alpha}_A$	axial thermal expansion coefficient
$\bar{\alpha}_T$	transverse thermal expansion coefficient

E_A^*	mixture rule estimate of axial Young's modulus
ν_A^*	mixture rule estimate of axial Poisson's ratio
k_T^*	mixture rule estimate of transverse bulk modulus
$\bar{\lambda}_A^*$	mixture rule estimate of axial shear modulus
$\bar{\alpha}_A^*$	mixture rule estimate of axial thermal expansion coefficient

0 parameters for unidirectional composites appearing in various formulae for the effective thermo-elastic constants

It should be noted, for isotropic fibres, matrix and transverse isotropic composite, respectively that

The transverse bulk modulus for a transverse isotropic composite is defined by

For an isotropic solid the bulk modulus is given by

NOTE : The effective properties of a unidirectional composite are denoted by *italic* symbols.

For orthotropic laminates :

- E_A axial Young's modulus (axial direction defined by principal loading direction)
 E_T transverse Young's modulus (in-plane of laminate normal to axial direction)
 E_t transverse Young's modulus (out-of-plane of laminate normal to axial direction)
 ν_A axial Poisson's ratio (in-plane)
 ν_a axial Poisson's ratio (out-of-plane)
 ν_t transverse Poisson's ratio (out-of-plane)
 $\bar{\lambda}_A$ axial shear modulus (in-plane)
 $\bar{e}_A$ axial shear coupling ratio
 $\bar{e}_T$ transverse shear coupling ratio

 $\bar{a}_A$ axial thermal expansion coefficient (in-plane)
 $\bar{a}_T$ transverse thermal expansion coefficient (in-plane)
 $\bar{a}_t$ transverse thermal expansion coefficient (out-of-plane)
 $\bar{a}_s$ thermal coefficient defining shear deformation

Other parameters :

- $\Delta T = T - T_0$ where T is current temperature and T_0 is the stress-free temperature
 $\bar{u}$ a parameter denoting that the quantity to which it is attached has a distribution of ply cracking

 $2h_1$ total thickness of all 90° plies in a cross-ply laminate
 $2h_2$ total thickness of all 0° plies in a cross-ply laminate
 $2h$ total thickness of laminate

 $\bar{\sigma}$ effective axial applied stress
 $\bar{\sigma}_T$ effective transverse applied stress (in-plane)
 $\bar{\sigma}_t$ transverse applied stress (out-of-plane)
 $\bar{\sigma}$ in-plane effective applied shear stress (in-plane)
 $\bar{\sigma}_0$ ply crack closure stress for uniaxial axial loading

 $\bar{a}_t^i$ transverse strain of undamaged laminate (out-of-plane)
 $\bar{a}$ effective axial strain of undamaged laminate
 $\bar{a}_T$ transverse strain of undamaged laminate (in-plane)
 $\bar{a}$ applied shear strain of undamaged laminate (in-plane)
 $\bar{a}_t(\bar{u})$ effective transverse strain of cracked laminate (out-of-plane)
 $\bar{a}(\bar{u})$ effective axial strain of cracked laminate
 $\bar{a}_T(\bar{u})$ uniform transverse strain for the cracked laminate (in-plane)
 $\bar{a}(\bar{u})$ applied shear strain of damaged laminate (in-plane)

 $2L$ distance between two neighbouring transverse cracks
 $\bar{\sigma}_0$
 $\bar{\sigma}_0$ laminate constants for inter-relationships between thermo-elastic constants
 $\bar{A}$ energy absorbed by ply crack formation
 $2\bar{a}_j$ fracture energy at the j^{th} potential crack site, corresponding to fracture energy in a monolithic laminate made from 90° plies subject to mode I deformation
 $\bar{\sigma}$ effective applied cracking stress

- $P(\dot{u})$ total energy of a damaged laminate
 $\tilde{A}(\dot{u})$ energy absorbed per unit volume by ply crack formation
 $g(\dot{u})$ Gibbs free energy per unit volume for cracked laminate
 $\bar{O}(\dot{u})$ parameter used to characterise state of laminate damage

3. Objectives of the methodology

The first objective of this guide is to provide recommended methods for predicting the effective thermo-elastic constants of fibre reinforced composites, in both unidirectional and laminated forms, from fibre and matrix properties, assuming that there is a good bond between fibre and matrix, and between the plies of laminates. For the case of laminates, predictions can be made using ply data predicted from fibre and matrix properties using suitable recommended models, or from experimentally measured data.

When laminated composites are loaded sufficiently, microstructural damage in the form of ply cracking occurs which modifies the effective thermo-elastic constants of the laminate. The second objective of the guide is to indicate how an assessment can be made of the changes in the effective thermo-elastic constants resulting from ply crack formation. In addition it is indicated how the thermo-elastic constants may be predicted as a function of the applied loading which may involve both in-plane biaxial and shear loading modes.

3.1 Applicable materials

The methodology can be applied to any fibre reinforced composite whose constituents can be regarded as deforming in a linear elastic manner, such as, thermosetting polymer composites and ceramic matrix composites. The methodology can be applied to monotonically loaded fibre reinforced thermo-plastic and metal composites only if matrix yield and creep are not significant. The methodology can also be applied to fatigue damaged fibre reinforced metal laminates to assess the effect on effective properties of ply cracks that have formed as a result of fatigue loading.

3.2 Assumptions of the methodology

The assumptions on which the methodology is based are :

- 1) for unidirectional composites the fibre and matrix are made of different materials (possibly anisotropic), and are perfectly bonded together everywhere on fibre/matrix interfaces. The cross-section of the fibres is assumed to be uniform and circular. The fibres are assumed to be uniformly distributed having a hexagonal packing structure leading to transverse isotropic composite properties.
- 2) laminates are symmetric and constructed of any number of plies that can be made of different anisotropic materials, and are perfectly bonded together at all points on the inter-laminar interfaces.
- 3) there exists a stress-free temperature T_0 at which unidirectional composites or laminates, when in an unloaded state, are free of any internal stresses. The temperature difference $\Delta T = T - T_0$ appears in the stress/strain relations and, when non-zero, leads to thermal residual stresses in the composite, even when unloaded.
- 4) the individual fibres and matrix of unidirectional composites, and the plies of laminates, deform as linear thermo-elastic anisotropic solids.
- 5) any ply cracks in the plies of laminates are assumed to be planar and fully developed (i.e. the entire cross-section of a ply is cracked).

4. Prediction of thermo-elastic constants for undamaged composites

Given the bewildering number of ply materials that can be made from the wide variety of available fibres and matrices, and the large number of laminate lay-ups that can be constructed from unidirectional plies, it is obvious that modelling undamaged composites from fibre and matrix properties is a key requirement that has in fact already been achieved and used extensively in design. The principal objective here is to indicate convenient methods for calculating the effective thermo-elastic constants of undamaged unidirectional composites, some of which are almost impossible to measure, for input into the modelling of laminate behaviour (including damaged laminates). Concentric cylinder models and variational techniques have been used extensively to predict the properties of unidirectional composites (see review by Hashin [1]).

Unidirectional composites are realistically modelled as transverse isotropic solids as fibre arrangements in real materials, while exhibiting non-uniform fibre spacings, are such that in the plane normal to the fibre direction the properties are more or less isotropic. Regular hexagonal arrays of fibres lead exactly to transverse isotropic properties, while square and diamond arrays clearly do not have such properties. Transverse isotropic solids are characterised by seven independent thermo-elastic constants which are selected to be :

E_A	axial Young's modulus
E_T	transverse Young's modulus
ν_A	axial Poisson's ratio
ν_T	transverse Poisson's ratio
$\dot{\lambda}_A$	axial shear modulus
$\dot{\alpha}_A$	axial thermal expansion coefficient
$\dot{\alpha}_T$	transverse thermal expansion coefficient.

Methods of estimating the values of these parameters for undamaged unidirectional composites will now be given.

4.1 Predictions using mixtures rule

Initial estimates of the axial modulus, Poisson's ratio, transverse bulk modulus, axial shear modulus and the axial thermal expansion coefficient can be obtained using the following rule of mixtures formulae for the special case of isotropic fibres and matrix

00

0

0

(4.

(4.

(4.

0

(4.

While it is not recommended

that these results are used for property predictions, the mixtures estimates of the above thermo-elastic constants do appear in more accurate formulae to be given in section 4.2. The values for the axial modulus predicted by (4.1) are in fact quite accurate.

4.2 Models for unidirectional composites

For unidirectional fibre reinforced composites, the concentric two cylinder model has been used extensively to predict more accurately the effective thermo-elastic constants in terms of the properties of the fibres and the matrix. Assuming that the fibres and matrix are both isotropic, the effective axial Young's modulus, the axial Poisson's ratio and the transverse bulk modulus are predicted by the concentric cylinders model to have the following values (see for example [2])

$$\begin{aligned} 0 & \\ 0 & \quad (4. \\ 0 & \quad (4. \\ 0 & \quad (4. \end{aligned}$$

where in (6-8)

$$0 \quad (4.$$

As the values
of E_A and 0 are
known from

(4.6) and (4.7) the value of $(1 - 0)/E_T$ can now be calculated using (4.8). As the unidirectional composite is being modelled as a transverse isotropic solid the following relationship holds

Clearly both E_T and 0 can be calculated once the value of $\hat{\imath}_T$ has been determined. Christensen and Lo [3] have developed a concentric cylinders model that can be used to estimate a value of the transverse shear modulus which lies between the bounds given by Hashin [1], and likely therefore to predict more accurate values. The Christensen and Lo model can, therefore, be used to calculate the transverse shear modulus $\hat{\imath}_T$ enabling values of E_T to be determined as described above. The details of the calculations are too complicated to be included here. The following expression given by Hashin [1, but beware of a misprint in his paper], leads to either an upper or lower bound value for $\hat{\imath}_T$, depending upon the values of $\hat{\imath}_f$, $\hat{\imath}_m$, k_f and k_m

$$0 \quad (4.1$$

The axial
shear modulus
 $\hat{\imath}_A$ is given [2]

by the formula

$$0 \quad (4.1$$

The axial and
transverse
thermal

expansion coefficients predicted by the model are given by

$$0 \quad (4.1$$

0

(4.1

The relations
(4.11 - 4.13) are
equivalent to
those given by
Hashin [1],

and Rosen and Hashin [4]. The formulae (4.11 - 4.13) are, however, preferred as they have the same desirable structure as the relations (4.6 - 4.8) presented by Hashin. If the fibre and/or matrix are transverse isotropic then Hashin has shown how the above formulae must be modified. Such a modification would be required when considering carbon fibres.

For composites where fibre coatings or reaction products (e.g. fibre reinforced titanium) are encountered, predictions can be made of the thermo-elastic constants using a multiple concentric cylinder model described in [5]. This model can deal with any number of transverse isotropic concentric layers.

4.3 Model for cross-ply laminates

Consider a multiple ply cross-ply laminate subject to triaxial loading. Symmetry is assumed about the mid-plane of the laminate. For the case when the symmetric laminate is undamaged the effective thermo-elastic constants of the multiple ply laminate are identical to those for an equivalent simple $[0_m/90_n]_s$ laminate. A label $i = 1$ will be used to denote parameters associated with the 90° ply of the equivalent laminate, and a label $i = 2$ will be used to denote parameters associated with the 0° ply of the equivalent laminate. The total thickness of the 90° plies is denoted by h_1 and the total thickness of the 0° plies is denoted by h_2 such that the total laminate thickness is $2h$ where $h = h_1 + h_2$. While the models described in section 4.2 assume that unidirectional plies are transverse isotropic materials characterised by seven independent thermo-elastic constants, the following analysis generalises the situation so that each ply of a laminate can be an orthotropic material characterised by twelve independent thermo-elastic constants.

Stress-strain-temperature relations

For the equivalent $[0_m/90_n]_s$ cross-ply laminate the stress-strain-temperature relations of the plies are assumed to be of the following linear form, for $i = 1, 2$

000

(4.1

where $\hat{\epsilon}_T$ and
 $\hat{\epsilon}_T$ are the axial
and transverse

(4.1

(4.1

in-plane strains that have the same value in the 0° and 90° plies respectively, and in the 0° and the 90° plies of the equivalent laminate. The quantities $\hat{\epsilon}_T$, $i = 1, 2$, appearing in (4.14) denote the corresponding through-thickness strains. The in-plane axial and transverse stresses are denoted by $\hat{\sigma}_T$, while the through-thickness stress has the same value $\hat{\sigma}_T$ in all the 0° and 90° plies of the actual laminate, and of the equivalent laminate. In (4.14 - 4.16), $\hat{\Delta}T$ is the temperature difference, defined by $\hat{\Delta}T = T - T_0$ where T is the current temperature of the material, and T_0 is the "manufacturing" temperature at which the strain is zero and the material is everywhere stress-free, with no internal or imposed external stresses and displacements.

The parameters E , ν and α denote the Young's modulus, Poisson's ratio and thermal expansion coefficient respectively. The thermo-elastic constants E , ν and α are allowed to have different values in the 0° and 90° plies of the equivalent laminate. The superscript i attached to the thermo-elastic constants indicates the type of ply to which they refer, i.e. the 90° ply of the equivalent laminate is denoted by $i = 1$ while the 0° ply is denoted by $i = 2$. Each ply has been assumed to be orthotropic so that twelve thermo-elastic constants are required to characterise linear behaviour. The upper case subscripts A and T are attached to axial and transverse thermo-elastic constants to denote that they refer to in-plane stresses and deformation while the corresponding lower case subscripts denote thermo-elastic constants that involve out-of-plane stresses and deformations. The 0° and 90° plies of the laminate are assumed to be made of the same material, but note that the form of the stress/strain relations (4.14 - 4.16) has assumed that the axial direction of each ply is oriented in the y-direction (independent of the fibre direction) so that for the 90° ply

where 0 are the thermo-elastic constants of the plies of the composite. For the 0° plies the following identifications must be made

Thermo-elastic constants for undamaged laminates

Define σ and σ_T to be, respectively, the effective axial stress and the in-plane transverse stress experienced by the laminate as a whole, so that

$$0$$

(4.1)

It has been shown [6] that the effective in-plane stress-strain relations for an undamaged laminate may be written

$$\begin{aligned}
 &0 \\
 &\text{where} \\
 &0 \\
 &0 \\
 &0
 \end{aligned}
 \tag{4.1}$$

$$\begin{aligned}
 &\text{and where} \\
 &00
 \end{aligned}
 \tag{4.2}$$

$$\begin{aligned}
 &\text{On making } \hat{a} \\
 &\text{and } \hat{a}_T \text{ the} \\
 &\text{subjects of the} \\
 &\text{equations} \\
 &(4.18), \text{ so that they are of the same form as equations (4.15) and (4.16), it has been shown [6]} \\
 &\text{that} \\
 &00 \\
 &\text{where} \\
 &0 \\
 &0 \\
 &0 \\
 &0
 \end{aligned}
 \tag{4.2}$$

$$\begin{aligned}
 &\text{It has also} \\
 &\text{been shown} \\
 &[6] \text{ that the} \\
 &\text{through-} \\
 &\text{thickness transverse modulus is given by}
 \end{aligned}
 \tag{4.2}$$

$$\begin{aligned}
 &\text{and the} \\
 &\text{through-} \\
 &\text{thickness} \\
 &\text{thermal} \\
 &\text{expansion} \\
 &\text{coefficient is} \\
 &\text{given by} \\
 &0
 \end{aligned}
 \tag{4.2}$$

$$\begin{aligned}
 &\text{The thermo-} \\
 &\text{elastic} \\
 &\text{constants} \\
 &\text{given by (4.26} \\
 &\text{- 4.30) characterise the thermo-elastic properties of an undamaged cross-ply laminate by} \\
 &\text{means of the following effective stress-strain relations :}
 \end{aligned}
 \tag{4.3}$$

000

(4.3

The

(4.3

corresponding

(4.3

shear stress-

strain relations are not given as they are not needed for subsequent developments.

5. Effects of ply cracking on thermo-elastic constants of cross-ply laminates

The first damage mode encountered when laminates are subject to multi-axial loading is the formation of ply cracks in plies whose fibre direction is transverse to the principal loading direction. For laminated systems, a methodology has been developed that is capable of predicting accurately the stress and displacement distributions at all points in a cross-ply laminate containing arrays of ply cracks. Such solutions are used to predict the dependence of the laminate properties, such as Young's moduli, Poisson's ratios and thermal expansion coefficients, on the degree of cracking, and more importantly, the dependence of these properties on the state of loading. Similar predictions can in principle be made for combined in-plane biaxial and shear loading modes (see section 7 concerning general laminates), as well as out-of-plane bending. The models are highly accurate in the sense that solutions developed have the following properties:

- The models retain all relevant stress components,
(*Shear-lag type theories certainly do not*)
- The models provide both stress and displacement fields,
(*Stress based variational methods usually do not*)
- The stress and displacement fields are such that the following are satisfied exactly:
 - equilibrium equations,
 - compatibility equations,
 - interfacial continuity conditions for both tractions and displacements,
 - external boundary conditions involving tractions,
 - all stress / strain relations, except for the axial equation.

However, when averaged through the thickness of each ply, the axial stress-strain relation can be satisfied exactly. This averaging approximation ensures that the stress and displacement field representations lead to stationary values of the Reissner energy functional [7] when carrying out a variational analysis. The models developed are thus of very high quality when assessed critically from a mechanics point of view. This means that engineers will be able to place great confidence in their predictions of behaviour. Furthermore, the high quality of the models has led to the development of a generalised framework [8] for the prediction of microstructural damage formation that leads to very simple relationships that are readily usable by the engineer. These relations apply to the macroscopic behaviour of laminates even though they arise from a detailed analysis of damaged laminates at the ply level.

5.1 Macroscopic effective stress-strain relations for a laminate

Consider a multiple-ply cross-ply laminate subject to multi-axial loading whose damage, in the form of stress-free ply cracks, is characterised by the damage parameter $\bar{u}$. This parameter is a single label that is used to represent a series of damage modes (and associated variables), for example the effective crack densities in each ply. It is assumed that the distribution of damage is uniform on average such that the effective stress/strain relations of the damaged laminate may be expressed in the usual form

000

where, for

specified

values of $\hat{\sigma}_t$, $\hat{\sigma}$,

$\hat{\sigma}_T$ and $\hat{\sigma}_T$, the parameters $\hat{\epsilon}_t(\bar{u})$, $\hat{\epsilon}(\bar{u})$ and $\hat{\epsilon}_T(\bar{u})$ are respectively the corresponding effective out-of-plane, axial and in-plane transverse strains of the damaged laminate, and where $E_A(\bar{u})$, $E_T(\bar{u})$, $E_t(\bar{u})$, $\nu_A(\bar{u})$, $\nu_a(\bar{u})$, $\nu_t(\bar{u})$, $\hat{\alpha}_A(\bar{u})$, $\hat{\alpha}_T(\bar{u})$ and $\hat{\alpha}_t(\bar{u})$ denote the effective thermo-elastic constants of the cracked laminate. In (5.1-5.3) the parameters $\hat{\sigma}$ and $\hat{\sigma}_T$ denote the effective applied axial and transverse stresses defined by (4.17), and $\hat{\sigma}_t$ denotes the imposed through-thickness stress that is uniform on the external faces of the laminate. It is assumed that the values of the thermo-elastic constants in (5.1-5.3) may be determined by detailed stress analysis for any distribution of ply cracks.

It has been shown [8], by considering the crack closure conditions for a laminate subject to uniaxial loading in the axial direction, that

0

where

0

and where A ,

B and C are

constants

calculated

from unidirectional properties as follows

The parameter k_1 appearing in (5.4) is a laminate constant that is independent of the damage parameter $\bar{u}$.

Similarly by considering the crack closure conditions for a laminate subject to uniaxial loading in the in-plane transverse direction it has been shown [8] that

0

where

0

The constant

k_2 is another

laminate

constant

independent

of the damage

parameter $\bar{u}$.

It follows from (5.4) and (5.6) that the following relations must also be valid

$$0 \quad (5.1)$$

where from
(5.5) and (5.7)

$$0 \quad (5.1)$$

The parameter k is a laminate constant that is independent of the damage parameter $\dot{u}$, and of the thermal expansion coefficients.

By considering the crack closure conditions for a laminate subject to uniaxial loading in the out-of-plane direction it has been shown [8] that

$$0 \quad (5.1)$$

where
0

(5.1)

The constant k_3 is thus another laminate constant independent of the damage parameter $\dot{u}$.

It follows from (5.4) and (5.10) that the following relations must also be valid

$$0 \quad (5.1)$$

where from
(5.5) and (5.11)
0

(5.1)

The parameter k_0 is another laminate constant that is independent of the damage parameter $\dot{u}$, and of the thermal expansion coefficients.

5.2 Useful independent inter-relationships

The relations (5.4), (5.6), (5.8), (5.10) and (5.12) are not of course all independent. It is, therefore, very useful to identify a set of independent inter-relationships. To achieve this objective it is convenient to introduce the parameter $\tilde{O}(\dot{u})$ defined by

0
 It should be
 noted that, as
 the axial
 modulus of a
 damaged laminate is nearly always measured as a function of crack density, the parameter $\bar{Q}$ is readily estimated from experimental data. Also, the dependence of the axial modulus on crack density is nearly always predicted by mathematical models of stress transfer, especially the elementary models based on shear lag theory. It can be shown, from (5.4), (5.8) and (5.12), that the following set of relations form the required independent set of inter-relationships
 0 (5.1)

0
 0 (5.1)
 0 (5.1)
 0 (5.1)
 00 (5.1)
 0 (5.2)
 0 (5.2)

In the above
 inter-relationships the parameters $\bar{Q}$ are constants for undamaged laminates defined by the relations (5.9), (5.13) and (5.5) respectively. On using (5.15 - 5.22), all the effective thermo-elastic constants of a damaged laminate can be calculated in terms of the damage dependent parameter $\bar{Q}(\bar{u})$ defined by (5.14).

The importance of these relationships must be emphasised. First of all, they enable an exact and simple means of fully characterising the effects of microstructural damage on the effective thermo-elastic constants of multiple ply cross-ply laminates, requiring only the value of the parameter $\bar{Q}(\bar{u})$ which depends upon the axial modulus that is frequently measured in practice, or can be estimated from a suitable model. Secondly, they enable an efficient method of calculating the effective thermo-elastic constants of damaged laminates using stress transfer models, avoiding a great deal of time consuming numerical computation when using state of the art models capable of predicting highly accurate results.

6. Predicting ply crack formation in multiple-ply cross-ply laminates

Consider a laminate that already contains cracks in the 90° plies characterised by the damage parameter $\bar{u}_1$. Then consider the simultaneous formation of new cracks such that the resulting damage state is characterised by the damage parameter $\bar{u}_2$. The locations of the cracks are assumed to be such that the overall deformation of the laminate is governed by the effective stress/strain equations (5.1 - 5.3). This means that the crack distribution is regarded as being sufficiently uniformly distributed for shear and out-of-plane bending deformations to be negligible. The new crack surfaces are assumed to form quasi-statically under conditions of fixed effective applied tractions and fixed temperature. From energy balance considerations and the fact that kinetic energy is never negative, the criterion for crack formation under these conditions has the form

0
where the
energy
absorbed in volume V of laminate by the formation of the new cracks is given by

0
In (6.2) the
parameter
 $\bar{A}(\bar{u})$ denotes

the energy absorbed in unit volume during the formation of new crack surfaces in the 90° plies that have led to the damage state $\bar{u}$. The energy absorbed is taken as the area of crack surface formed multiplied by the fracture energy for transverse cracking. The corresponding change of Gibbs free energy in the region V of the laminate is

0
The Gibbs free
energy is used
in (6.1)

because the applied stresses and temperature are being regarded as the independent state variables. If external displacements were specified rather than stresses then the Helmholtz free energy would replace the Gibbs free energy.

In (6.3) $g(\bar{u})$ denotes the Gibbs free energy per unit volume when the damage in the laminate is characterised by $\bar{u}$. It follows from (6.1 - 6.3) that crack formation associated with the change in damage state from $\bar{u}_1$ to $\bar{u}_2$ is governed by the inequality

0
where P is the
total energy
defined by

0
It has been
shown [8] that

0
where σ_0 is the
ply crack
closure stress

when the laminate is subject to uniaxial loading in the axial direction normal to the ply cracks. The applied stress dependence appears as a perfect square involving a linear combination of the applied stresses.

It has also been shown [8] that

000

Thus, on using

(5.9), (5.13)

and (5.14) the

result (6.6) may be written in either of the following three equivalent forms

000

It is

emphasised

that the

exceedingly

simple results

(6.10 - 6.12) are exact, and that they take full account of the effect of multi-axial loading and of the effect of thermal residual stresses. The results involve only macroscopic properties defined at the laminate level even though they have been derived from a detailed consideration of the stress and deformation distributions in a cracked laminate at the ply level. The results are based on the single assumption of damage homogeneity where damage formation in the laminate is such that the stress/strain relationships (5.1 - 5.3) are always satisfied.

6.1 Ply crack initiation criteria

For crack initiation $\dot{u}_1 = \dot{u}_0$ (the value of $\dot{u}$ for the undamaged state) and $\dot{u}_2 = \dot{u}$ with the result that the use of (6.2), (6.4) and (6.10 - 6.12) leads to the following three equivalent forms for the crack initiation criterion

0

or

0

or

0

It is

emphasised

that the

approach

described does

not provide any information that indicates how the thermo-elastic constants depend upon the damage parameter $\dot{u}$. A detailed stress analysis is required to provide this information (see [9]).

6.2 Progressive ply cracking

It is useful to discuss progressive cracking in a length $2L$ of laminate where the damage parameter $\dot{u}$ is defined by $\{L_1, L_2, \dots, L_n\}$ where n is the number of cracks in length $2L$ of the 90° plies and where $L_i, i = 1 \dots n$ denote the distances between neighbouring cracked planes. For this case the energy absorption per unit volume for length $2L$ of laminate is given by

0
 where $2h_1$ is
 the total
 thickness of all
 90° plies in the
 laminate having total thickness $2h$, and where N is the number of potential cracking sites in
 90° plies which are ordered in a regular way, e.g. from top to bottom in the plies which are
 taken in order from the centre of the laminate to the outside, symmetry about the mid-plane
 of the laminate being assumed. The quantity $2\tilde{a}_j$ is the fracture energy for the j^{th} potential
 cracking site of the 90° plies. The parameters $\tilde{a}_j$ describe the crack pattern in the laminate
 such that

(6.1)

It follows from
 this definition
 that the
 number of
 cracks in
 length $2L$ of
 laminate is
 given by
 0

(6.1)

For
 progressive
 cracking the
 parameters $\tilde{u}_1$ and $\tilde{u}_2$ appearing in (6.2) are taken to be
 0

(6.1)

It has been
 shown [8] that
 the criterion for progressive cracking may be written in the simple form
 0

(6.2)

It must be
 emphasised
 that the result
 (6.20) takes
 full account of
 anisotropy and thermal residual stresses, and is valid for multi-axial loading conditions.
 The result is exact and simple in form. It is concluded that accurate stress transfer models
 are needed only to estimate thermo-elastic constants of a cracked laminate. All other aspects
 of crack formation can be dealt with using the general framework described above.

When using (6.20) to investigate crack formation the fracture energies $2\tilde{a}_j$ are taken at
 random from a statistical distribution of fracture energies (assumed to be normal) and then
 allocated at random to the various potential fracture sites in the 90° plies. The value of the
 applied stresses are then regarded as slowly increasing until the fracture condition (6.20)
 can be met at just one potential fracture site in the laminate. This means that crack
 formation must be investigated at all sites in order to determine the location of the first

crack to form. This procedure has to be repeated for every new crack that forms during loading.

6.3 An effective ply cracking stress

The results (6.6 - 6.9) indicate that for multi-axial loading it will be useful to introduce an effective ply cracking stress denoted by s and defined by

0 (6.2)

It has recently
been shown
numerically

(unpublished work) that the effective stress defined by (6.21) also appears in a condition determining the steady state delamination from a transverse crack in a cross-ply laminate described in [6]. This remarkable fact requires further investigation. It follows from (6.6) that

0 (6.2)

where from
(5.14) and (6.8)

0 (6.2)

The relation
(6.20) for
progressive

ply cracking may then be written in the following simple and very useful form

0 (6.2)

and, when
applied to the
prediction of
ply crack
initiation,

(6.24) reduces
to the form

0 (6.2)

It should be
noted from [8]
that at the
point of ply
crack closure

0 (6.2)

7. Effects of ply cracking on thermo-elastic constants of general laminates

While a great deal of modelling work has been carried out for simple cross-ply laminates subject to uniaxial loading, much less work has been carried out modelling ply cracking in general symmetric laminates where $\pm\bar{e}$ and quasi-isotropic geometries are of particular importance. Very little work has been carried out on complex loading modes involving combined in-plane biaxial and shear loading, and out-of-plane bending. Recent progress has been made addressing these complex issues of direct relevance to engineering design. In particular, a new rational predictive framework [10] for general symmetric laminates subject to general in-plane loading has been developed. The approach assumes that the applied through-thickness stress is always zero, and that linear elastic behaviour in each ply leading to non-linear behaviour of the laminate caused by ply crack formation. Assuming that deformations are linear elastic at the microstructural level and that ply crack formation is homogeneous when viewed at the macroscopic level, the approach is exact and can be used with confidence when applied to laminated composites. The framework is again based on energy principles where for quasi-static conditions the energy available for crack formation is balanced by the energy needed to form new ply crack surfaces. The framework is developed in terms of the effective thermo-elastic properties of the damaged laminate, and it has been shown that the criteria for ply crack initiation and progressive formation can be written down simply and solely in terms of parameters that are defined at the macroscopic laminate level even though the methodology has analysed behaviour at the individual ply level, and at the fibre/matrix level for conditions of perfect fibre/matrix bonding. The framework is derived without having to introduce a stress transfer model that accounts for localised stress redistribution in the laminate arising from ply crack formation. Such a model [11] is needed only to calculate the effective thermo-elastic constants of a general symmetric laminate containing ply cracks.

A description of the properties and characteristics of a recently developed high quality stress transfer model has been given in [11] together with its application to the prediction of the dependence of the effective thermo-elastic constants of a laminate on ply crack density, including accounting for non-uniform crack distributions. The stress transfer model is based on stress and displacement fields, defined at all points within a general symmetric laminate subject to general in-plane loading, that satisfy exactly:

- equilibrium equations,
- compatibility equations,
- interfacial continuity conditions for both tractions and displacements,
- external boundary conditions involving tractions,
- all stress/strain relations except for : *in-plane axial, transverse and shear equations*.

The in-plane stress/strain relations and displacement boundary conditions cannot be satisfied by the representation but can be satisfied if they are averaged through the thickness of each ply, or of each ply element when using ply refinement techniques to increase the accuracy of predictions. The resulting averaged equations are such that if the representation were to be used in a variational calculation based on the Reissner energy functional [7] denoted by J , then the satisfaction of the averaged stress/strain relations is sufficient to ensure that $\delta J = 0$, indicating that the model is one of high quality.

7.1 Macroscopic effective stress-strain relations for a laminate

Consider any symmetric multiple-ply laminate whose damage is characterised by the parameter $\bar{u}$. It is assumed that the distribution of damage in each ply is effectively uniform at the macroscopic level such that the effective stress-strain relations of the damaged laminate may be expressed in the same form as that for an uncracked laminate, namely

000

where ΔT is

the

temperature

difference,

defined by $\Delta T = T - T_0$ where T is the current temperature of the material, and T_0 is the "manufacturing" temperature at which the strain is zero and the material is everywhere stress-free, with no internal or imposed external stresses and displacements. In (7.1 - 7.3) the parameters $\bar{\sigma}$, $\bar{\sigma}_T$ and $\bar{\sigma}$ denote the in-plane effective applied axial, transverse and shear stresses. For specified values of $\bar{\sigma}$, $\bar{\sigma}_T$, $\bar{\sigma}$ and ΔT , the quantities $\bar{\epsilon}(\bar{u})$, $\bar{\epsilon}_T(\bar{u})$ and $\bar{\epsilon}(\bar{u})$ are respectively the in-plane axial, transverse and shear strains of the damaged laminate, and $E_A(\bar{u})$, $E_T(\bar{u})$, $\nu_A(\bar{u})$, $\bar{\epsilon}_A(\bar{u})$ and $\bar{\epsilon}_T(\bar{u})$ denote the corresponding effective thermo-elastic constants of the cracked laminate. The parameters $\bar{\epsilon}_A(\bar{u})$ and $\bar{\epsilon}_T(\bar{u})$ are ratios indicating the degree of shear coupling. These parameters are zero for the special case of cross-ply laminates. The parameter $\bar{\epsilon}_s(\bar{u})$ is a thermal expansion coefficient governing the amount of shear strain that can arise when the temperature is changed.

It is understood that the values of the thermo-elastic constants in (7.1 - 7.3) may be determined by detailed stress analysis for any distribution of fully developed ply cracks. Indeed, calculations using the methods described in [11] indicate that for the case of uniform crack densities, stress-strain relations of the form (7.1 - 7.3) are obeyed. When the laminate is undamaged the thermo-elastic constants are written without showing any dependence on the damage parameter $\bar{u}$: thus E_A denotes the axial Young's modulus for an undamaged laminate, and similarly for the other thermo-elastic constants.

It follows from (7.3) that

0

Substitution in

(7.1) and (7.2)

then leads to

the following

reduced stress-strain equations that

are of the same form as those valid for in-plane biaxial loading modes

00

where 0 can be

interpreted as

the axial and

transverse

strains for a cracked laminate that is constrained so that the shear strain is zero, and where

(7.

(7.

(7.

(7.

(7.

(7.

00
0
00

(7.
(7.
(7.1
(7.1

7.2 Useful independent inter-relationships

It has been shown [10] that the following relations can be satisfied for ANY state of cracking in the 90° plies only if the following inter-relationships are satisfied of which three are independent

0

(7.1

where K is a constant that depends only on the geometry and properties of the plies in the laminate. It has been shown from extensive numerical solutions using methods described in [11] that the inter-relationships (7.12) are indeed satisfied for all stages of uniform cracking. It follows from (7.12) that the following inter-relationships are also satisfied

00

(7.1
(7.1

where α and β are other constants that also depend upon the geometry and properties of the plies in the laminate.

It is useful to investigate ply crack closure for the special case of axial loading such that the shear strain $\tilde{\alpha} = 0$ and the transverse stress $\sigma_T = 0$. For this case, deformation under a constrained shear strain will in most cases lead to a non-zero shear stress that is needed to maintain the zero shear strain. Thus the loading case being considered is not one of uniaxial loading. At the point of crack closure α and it follows from (7.5) that $\alpha = 0$ where, on using (7.13),

0

(7.1)

It is also
useful to
investigate ply
crack closure

for the special case of transverse loading such that the shear strain $\tilde{\alpha} = 0$ and the axial stress $\tilde{\sigma} = 0$. For this case deformation under a constrained shear strain will in most cases lead to a non-zero shear stress that is needed to maintain the zero shear strain. Thus the loading case being considered is again not one of uniaxial loading. At the point of crack closure 0 and it follows from (7.6) that 0 where, on using (7.14),

0

(7.1)

It has been
shown [10]
that

0

(7.1)

Expressions in
terms of ply
properties for
the laminate

parameters 0 are derived in [10, Appendix A], and are given by

0

(7.1)

0

(7.1)

0

(7.2)

It is worth
noting that if 0
is *measured*
during damage
development

then in principle the values of the parameters 0 can be deduced using the relations (7.12 - 7.14) which are expected to be exact within the assumptions of linear elasticity, stress-free cracks and homogeneous damage at the macroscopic scale.

8. Predicting damage formation in general laminates

For general symmetric laminates the relations (6.1 - 6.5), derived for application to cross-ply laminates, remain valid. It has been shown [10] that for the case of a general symmetric laminate the total energy defined by (6.5) may be expressed

$$0 \quad (8.1)$$

where σ_0 are defined by

(7.5). It has been shown

[10] that the result (8.1) can

also be written in the equivalent form

$$0 \quad (8.2)$$

where σ_0 are defined by

(7.6).

The results

(8.1) or (8.2),

when used in conjunction with the equation (6.4) governing ply crack formation in a general symmetrical laminate, lead to exceedingly simple results that involve only macroscopic properties defined at the laminate level. Furthermore, the results take full account of the effects of combined biaxial and shear loading and of the thermal residual stresses. The results are based on the assumptions that damage formation in the laminate is such that the stress-strain relationships (7.1 - 7.3) are always satisfied and that the ply cracks are stress-free.

8.1 Ply crack initiation criteria

For crack initiation $\dot{u}_1 = \dot{u}_0$ (the value of $\dot{u}$ for the undamaged state) and $\dot{u}_2 = \dot{u}$ with the result that the use of (6.4) and (8.1) leads to the criterion

$$0 \quad (8.3)$$

or

equivalently

on using (8.2)

where σ_0 and

similarly for the

other thermo-

elastic

constants. It is

emphasised that

the approach described does not provide any information that indicates how the thermo-elastic constants depend upon the damage parameter $\dot{u}$. The detailed stress analysis described in [11] is needed to obtain this information.

8.2 Progressive ply cracking

For general symmetric laminates subject to general in-plane loading, oriented so that the

fibres of the plies in which ply cracks may form are aligned in the 90° direction, the energy absorbed as a result of ply cracking can be estimated using the relations (6.16 - 6.19) derived for cross-ply laminates. It has been shown [10] that the inter-relationships (7.12), also valid for non-uniform crack spacings (see [10]), may be generalised to the form

$$0 \quad (8.1)$$

for any values of the damage parameters $\bar{u}_1$ and $\bar{u}_2$. On using (6.4), (8.1) and (8.5) it can be shown that the criterion for progressive cracking may be written in the compact form

$$0 \quad (8.2)$$

which is consistent with the initiation condition (8.3)

when $\bar{u}_2 = \bar{u}$ and $\bar{u}_1 = 0$. It must be emphasised that the result (8.6) takes full account of anisotropy and thermal residual stresses, and is valid for combined biaxial and shear loading conditions. The result is exact and simple in form. It is concluded that stress transfer models are needed only to estimate thermo-elastic constants of a cracked laminate. All other aspects of crack formation can be dealt with using the general framework described here. It is also emphasised that the criterion (8.6) is valid *only* if new cracks form for fixed values of $\bar{\sigma}$, $\bar{\sigma}_T$, $\bar{\delta}$ and $\ddot{A}T$.

It should be noted that the relation (8.6) valid for general symmetric laminates reduces to the (6.20) for the special case of cross-ply laminates. This is because, in the case of cross-ply laminates, the macroscopic applied shear strains are taken to be zero, and the shear coupling parameters $\ddot{e}_A$ and $\ddot{e}_T$ always zero so that from (7.5) 0, and from (7.7 - 7.11) 0, and similarly for the other thermo-elastic constants.

The criteria (8.3) and (8.6), derived for initiation and progressive ply cracking respectively, would have rather complicated forms if expressed directly in terms of $\bar{\sigma}$, $\bar{\sigma}_T$, $\bar{\delta}$ and $\ddot{A}T$. The L.H.S. of (8.3) and (8.6) would lead to quadratic forms in the variables $\bar{\sigma}$, $\bar{\sigma}_T$, $\bar{\delta}$ and $\ddot{A}T$ that are difficult to solve in general. To simplify the problem, the effective applied stresses are now assumed to be proportional so that during loading

$$0 \quad (8.3)$$

where A and B are given

values.

On substituting in (8.6) in terms of $\hat{\sigma}$, $\hat{\sigma}_T$, $\hat{\sigma}$ and $\hat{A}_T$ for the reduced axial strain and the shear strain using (7.3) and (7.5) together with the relations (8.7), the criterion (8.6) may be expressed in the following quadratic form involving only the axial applied stress $\hat{\sigma}$

$$a\hat{\sigma}^2 + b\hat{\sigma} + c = 0 \quad (8.1)$$

where the coefficients a ,

b and c (dependent on the loading parameters A and B) can be defined from the L.H.S. of (8.6), which is denoted by $f(\hat{\sigma})$, using the following relations

$$a = \frac{1}{2} \left(\frac{\partial f}{\partial \hat{\sigma}} \right)_{\hat{\sigma}=0}, \quad b = f(0), \quad c = \frac{1}{2} \left(\frac{\partial f}{\partial \hat{\sigma}} \right)_{\hat{\sigma}=\hat{\sigma}_T} \quad (8.1)$$

Let $\hat{\sigma}^+$ and $\hat{\sigma}^-$

be the roots of

the quadratic equation

$$a\hat{\sigma}^2 + b\hat{\sigma} + c = 0 \quad (8.1)$$

The solutions

are clearly given by

$$\hat{\sigma} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (8.1)$$

If $b^2 < 4ac$ then

the values

given by (8.13)

are complex

and clearly ply cracking cannot occur. If $b^2 \geq 4ac$ then ply cracking occurs and predictions to date indicate that there is only one positive root which is the required solution. The results given by (8.7 - 8.13) can also be applied to the case of ply crack initiation governed by (8.3) instead of (8.6). The coefficients a , b and c would of course have different values to those used when predicting progressive ply cracking.

When using (8.6) to investigate crack formation the fracture energies $2\tilde{a}_i^{(90)}$ are taken at random from a statistical distribution of fracture energies (normal assumed) and then allocated at random to the various potential fracture sites in the 90° plies. The values of the applied stresses are then regarded as slowly increasing until the fracture condition (8.6) can be met at one of the potential fracture sites in the laminate. This means that crack formation, for every stage of loading, must be investigated at all remaining potential cracking sites in order to determine the location of the next crack to form. This procedure, carried out using a computer, has to be repeated for every new crack that forms during loading.

9. Conclusions

- 1) The generalised rational framework for predicting ply crack initiation and growth in laminated composites is an excellent basis for the development of new design methods that take proper account of the micro-mechanisms of damage formation because the approach is exact within the assumptions made and the methodology rigorously bridges the gap between ply related phenomena (e.g. ply cracking) and the macroscopic laminate behaviour of interest to engineering design.
- 2) Detailed stress transfer models are accurate and are needed only to predict values for the thermo-elastic constants as a function of geometry and crack distribution.
- 3) Laminate predictions can be made from basic fibre and matrix data offering significant potential savings in development costs for composite materials, and for the design and testing of composite components.
- 4) Because of their robustness and accuracy, the micro-mechanical models described have very good potential for being used seriously in the design of both composite materials, and of components made of these materials.
- 5) The methods [8-11] can be used to predict the multi-axial non-linear behaviour of laminates when ply cracking occurs. This leads to the possibility of replacing severe ply-discount strategies by less drastic and more realistic predictions that take account of ply crack initiation and the early stages of degradation in composite properties arising from the progressive formation of ply cracking. Such methods have potential for integration into FEA codes for application to the prediction of the initiation and progressive formation of ply cracks in the non-uniform stress fields that arise in composite components.

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