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multiple-ply cross-ply laminates subject to biaxial loading
I : Development of model**

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ABSTRACT

Stress transfer is considered due to transverse cracking in a multiple-ply cross-ply laminate subject to biaxial loading. A representation for the stress and displacement fields in the laminate, valid for generalised plane strain conditions, is derived that takes account of the presence of thermal residual stresses. The interfaces between plies in the laminate are assumed to be perfectly bonded, even in the neighbourhoods of the transverse cracks. The representation satisfies the equilibrium equations, the interfacial boundary conditions and all stress-strain relations apart from the axial relations that are satisfied in an average sense (i.e. the average of each axial stress-strain relation is satisfied exactly on averaging through the ply thickness). The solution method involves the operation of sets of recurrence relations, and reduces the problem to that of solving a system of fourth order ordinary differential equations that are solved by numerical methods. The technique allows the checking of the satisfaction of the system of differential equations and both the boundary and interface conditions. The many solutions obtained to date (to be published) indicate that the solution method is highly accurate and robust as indicated by its successful application to GRP and CFRP cross-ply laminates for a range of cross-ply laminate lay-ups.

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1. INTRODUCTION

Cross-ply laminates are prone to the formation of transverse cracks when subjected to loading. Micro-cracking leads to a deterioration in mechanical properties, and if localised in regions of stress concentrations in composite structures, stress can be transferred from regions of stress concentration to other parts of the structure. This macroscopic stress transfer process often leads to composite structures whose strength exceeds that which would be predicted from laboratory test data obtained for homogeneous deformation states. In order to understand fully such macroscopic stress transfer processes in composite laminates it is first necessary to be able to predict the stress and displacement distributions in cracked laminates. These distributions can then be used to predict the dependence of the effective thermoelastic constants on crack density.

A great deal of attention has been devoted in the literature to developing models of stress transfer in micro-cracked 0° - 90° - 0° cross-ply laminates based on shear-lag approaches (e.g. [1, 2]) and on variational techniques (e.g. [3-9]). Very little attention (e.g. [10, 11]) has been devoted to developing stress transfer models for micro-cracked laminates that are affected both by the thermal residual stresses that arise in the laminate as a result of the manufacturing process, and by any transverse applied loads that can either increase or decrease the likelihood of micro-cracking in the transverse plies.

The objective of this report is to develop a generalised plane strain micromechanical model of stress transfer derived for 0° - 90° - 0° laminates to biaxially loaded multiple-ply cross-ply laminates, and develop a method for the prediction of the dependence of the thermoelastic constants on crack density (assumed uniform). Perfect bonding is assumed between the plies and full account is to be taken of the effects of thermal residual stresses. The stress transfer model is based on the fundamental assumption that the axial stresses in 0° and 90° plies are independent of the through-thickness coordinate. This is effectively the only assumption of the stress-transfer model that leads to the following characteristics of the solution,

- * the equilibrium equations are satisfied exactly,
- * the interface conditions for perfect bonding are satisfied exactly,
- * all stress-strain relations are satisfied exactly apart from the axial relations that are satisfied in an average sense (averaging in the through-thickness direction across each of the plies),
- * all external traction boundary conditions are satisfied exactly, and applied displacement conditions are satisfied in an average sense,
- * the problem reduces to solving a system of simultaneous fourth order ordinary differential equations.

It has been shown by Pagano [12] that the generalised plane strain model of stress transfer for a 0° - 90° - 0° laminate leads to stationary values of the Reissner energy functional [13] so that the stress and displacement distribution derived from the model would be consistent with that resulting from a corresponding variational calculation. This property of the generalised plane strain representation also applies for multiple-ply cross-ply laminates. Thus the stress transfer model to be considered in this report is the best that can be developed

based upon the single fundamental assumption that axial stresses in each ply are independent of the through-thickness coordinate. The combination of this property of the model together with layer refinement techniques means that virtually exact solutions can be obtained to stress transfer problems arising when transverse cracks form in multiple-ply cross-ply laminates.

2. GEOMETRY AND BASIC EQUATIONS

Geometry

The problem under consideration concerns the generalised plane strain deformation of a symmetric multi-layered cross-ply laminate constructed of $2N+2$ perfectly bonded plies. As symmetry in the through-thickness direction is assumed, it is necessary to consider only the right hand set of $N+1$ plies as shown in Fig.1. A global set of rectangular Cartesian coordinates are chosen having the origin at the centre of the laminate as shown in the figure. The y-direction defines the longitudinal or axial direction, the z-direction defines the transverse direction and the x-direction defines the through-thickness direction. The locations of the N interfaces are specified by $x = x_i$, $i = 1 \dots N$. The mid-plane of the laminate is specified by $x = x_0 = 0$ and the external surface by $x = x_{N+1} = h$ where $2h$ is the total thickness of the laminate. The thickness of the i^{th} ply is denoted by $h_i = x_i - x_{i-1}$. The orientation of the i^{th} ply is specified by the angle θ_i between the fibre direction of this ply and the y-axis. For cross-ply laminates $\theta_i = 0^\circ$ or 90° .

The laminate is loaded on the edges at $y = \pm L$ and $z = \pm W$ by the application of known displacements in the z and y directions applied to one or more uncracked plies. Stress, strain and displacement components, together with material properties associated with the i^{th} layer are denoted by a superscript or subscript i .

Equilibrium equations

The laminate is assumed to be in equilibrium, so that the following equations must be satisfied everywhere in the laminate

$$\frac{\partial \sigma_{xx}^i}{\partial x} + \frac{\partial \sigma_{xy}^i}{\partial y} + \frac{\partial \sigma_{xz}^i}{\partial z} = 0 , \quad (1)$$

$$\frac{\partial \sigma_{xy}^i}{\partial x} + \frac{\partial \sigma_{yy}^i}{\partial y} + \frac{\partial \sigma_{yz}^i}{\partial z} = 0 , \quad (2)$$

$$\frac{\partial \sigma_{xz}^i}{\partial x} + \frac{\partial \sigma_{yz}^i}{\partial y} + \frac{\partial \sigma_{zz}^i}{\partial z} = 0 . \quad (3)$$

Strain-displacement relations

The displacements in the x, y and z directions are denoted by u, v and w respectively, so that the strains can be calculated as follows

$$\epsilon_{xx} = \frac{du}{dx}, \quad \epsilon_{xy} = \epsilon_{yx} = \frac{1}{2} \left(\frac{du}{dy} + \frac{dv}{dx} \right), \quad (4)$$

$$\epsilon_{yy} = \frac{dv}{dy}, \quad \epsilon_{yz} = \epsilon_{zy} = \frac{1}{2} \left(\frac{dv}{dz} + \frac{dw}{dy} \right), \quad (5)$$

$$\epsilon_{zz} = \frac{dw}{dz}, \quad \epsilon_{zx} = \epsilon_{xz} = \frac{1}{2} \left(\frac{dw}{dx} + \frac{du}{dz} \right). \quad (6)$$

Stress-strain-temperature relations

For a cross-ply laminate the stress-strain-temperature relations take the form, for $i = 1 \dots N+1$

$$\epsilon_{xx}^i = \frac{1}{E_t^i} \sigma_{xx}^i - \frac{\nu_a^i}{E_A^i} \sigma_{yy}^i - \frac{\nu_t^i}{E_T^i} \sigma_{zz}^i + \alpha_t^i \Delta T, \quad (7)$$

$$\epsilon_{yy}^i = -\frac{\nu_a^i}{E_A^i} \sigma_{xx}^i + \frac{1}{E_A^i} \sigma_{yy}^i - \frac{\nu_A^i}{E_A^i} \sigma_{zz}^i + \alpha_A^i \Delta T, \quad (8)$$

$$\epsilon_{zz}^i = -\frac{\nu_t^i}{E_T^i} \sigma_{xx}^i - \frac{\nu_A^i}{E_A^i} \sigma_{yy}^i + \frac{1}{E_T^i} \sigma_{zz}^i + \alpha_T^i \Delta T, \quad (9)$$

$$\epsilon_{xy}^i = \frac{\sigma_{xy}^i}{2\mu_a^i}, \quad \epsilon_{xz}^i = \frac{\sigma_{xz}^i}{2\mu_t^i}, \quad \epsilon_{yz}^i = \frac{\sigma_{yz}^i}{2\mu_A^i}, \quad (10)$$

where ΔT is the temperature difference, defined by $\Delta T = T - T_0$ where T is the current temperature of the material, and T_0 is the "manufacturing" temperature at which the strain is zero and the material is everywhere stress-free, with no internal or imposed external stresses and displacements. The parameters E , ν , μ and α denote the Young's modulus, Poisson's ratio, shear modulus and thermal expansion coefficient respectively. The thermoelastic constants E , ν , μ and α are allowed to have different values in each ply of the laminate. The superscript i attached to the thermoelastic constants indicates the number of

the ply to which they refer. Each ply is assumed to be orthotropic so that twelve thermoelastic constants are required to characterise linear behaviour. The upper case subscripts A and T are attached to axial and transverse thermoelastic constants to denote that they refer to in-plane stresses and deformation while the corresponding lower case subscripts denote thermoelastic constants that involve out-of-plane stresses and deformations.

3. STRESS AND STRAIN DISTRIBUTIONS FOR AN UNDAMAGED LAMINATE

Consider an undamaged multi-layered laminate subject to a biaxial in-plane loading with an effective longitudinal stress and strain of σ and ϵ respectively, and effective transverse stress and strain of σ^* and ϵ^* respectively. It is assumed that the external surfaces of the laminate are stress free, and that the laminate is thin enough for the stress component σ_{xx} to be everywhere zero for an undamaged laminate. It then follows that

$$\sigma_{xx}^i \equiv \sigma_{xy}^i \equiv \sigma_{xz}^i \equiv \sigma_{yz}^i \equiv 0, \quad i = 1 \dots N+1. \quad (11)$$

Denoting the longitudinal and transverse stresses experienced by the i^{th} ply to be σ_i and σ_i^* respectively, equations (8) and (9) reduce to, for $i = 1 \dots N+1$,

$$\epsilon = \frac{1}{E_A^i} \sigma_i - \frac{\nu_A^i}{E_A^i} \sigma_i^* + \alpha_A^i \Delta T, \quad (12)$$

$$\epsilon^* = -\frac{\nu_A^i}{E_A^i} \sigma_i + \frac{1}{E_T^i} \sigma_i^* + \alpha_T^i \Delta T, \quad (13)$$

where ϵ and ϵ^* are respectively the axial and transverse in-plane strains that are experienced by all the plies in the laminate. Let σ and σ^* be, respectively, the corresponding effective axial and transverse stresses experienced by the laminate as a whole. From mechanical equilibrium, it can be deduced that

$$\sigma = \frac{1}{h} \left(\sum_{i=1}^{N+1} h_i \sigma_i \right), \quad \sigma^* = \frac{1}{h} \left(\sum_{i=1}^{N+1} h_i \sigma_i^* \right). \quad (14)$$

Inverting equations (12) and (13) to obtain the ply stresses in terms of ϵ and ϵ^* leads to

$$\sigma_i = \hat{E}_A^i \left(\epsilon + \nu_A^i \frac{E_T^i}{E_A^i} \epsilon^* - \alpha_A^i \Delta T \right), \quad (15)$$

$$\sigma_i^* = \hat{E}_T^i \left(\nu_A^i \epsilon + \epsilon^* - \alpha_T^i \Delta T \right), \quad (16)$$

where

$$\frac{1}{\hat{E}_A^i} = \frac{1}{E_A^i} \left(1 - (v_A^i)^2 \frac{E_T^i}{E_A^i} \right), \quad \alpha_A^i = \alpha_A^i + v_A^i \frac{E_T^i}{E_A^i} \alpha_T^i, \quad (17)$$

$$\frac{1}{\hat{E}_T^i} = \frac{1}{E_T^i} \left(1 - (v_A^i)^2 \frac{E_T^i}{E_A^i} \right), \quad \alpha_T^i = \alpha_T^i + v_A^i \alpha_A^i. \quad (18)$$

Substituting these expressions for σ_i and σ_i^* into (14) enables the effective stresses to be expressed in terms of the effective strains as follows

$$\sigma = A\varepsilon + B\varepsilon^* - P\Delta T, \quad (19)$$

$$\sigma^* = B\varepsilon + C\varepsilon^* - Q\Delta T, \quad (20)$$

where

$$A = \frac{1}{h} \left(\sum_{i=1}^{N+1} h_i \hat{E}_A^i \right), \quad B = \frac{1}{h} \left(\sum_{i=1}^{N+1} h_i v_A^i \hat{E}_T^i \right), \quad C = \frac{1}{h} \left(\sum_{i=1}^{N+1} h_i \hat{E}_T^i \right), \quad (21)$$

$$P = \frac{1}{h} \left(\sum_{i=1}^{N+1} h_i \hat{E}_A^i \alpha_A^i \right), \quad Q = \frac{1}{h} \left(\sum_{i=1}^{N+1} h_i \hat{E}_T^i \alpha_T^i \right). \quad (22)$$

On making ε and ε^* the subjects of the equations (19) and (20), so that they are of the same form as equations (12) and (13), it can be shown that

$$\varepsilon = \frac{1}{E_A} \sigma - \frac{v_A}{E_A} \sigma^* + \alpha_A \Delta T, \quad (23)$$

$$\varepsilon^* = -\frac{v_A}{E_A} \sigma + \frac{1}{E_T} \sigma^* + \alpha_T \Delta T, \quad (24)$$

where

$$E_A = A - \frac{B^2}{C}, \quad E_T = C - \frac{B^2}{A}, \quad v_A = \frac{B}{C}, \quad (25)$$

$$\alpha_A = \frac{P - v_A Q}{E_A}, \quad \alpha_T = \frac{1}{E_T} \left(Q - v_A \frac{E_T}{E_A} P \right). \quad (26)$$

Clearly, E_A and E_T are, respectively, the axial and transverse Young's moduli of the undamaged laminate as a whole, α_A and α_T are the axial and transverse thermal expansion coefficients for the laminate, and v_A is the axial Poisson's ratio for the laminate.

Note that the in-plane displacements for the undamaged laminate, denoted by v_i and w_i respectively, are simply

$$v_i = \varepsilon y, \quad w_i = \varepsilon^* z, \quad i = 1 \dots N+1. \quad (27)$$

4. GENERALISED PLANE STRAIN

When analysing stress transfer in laminates, generalised plane strain conditions are imposed so that the displacements u , v and w in the i^{th} ply of the laminate are of the form

$$u_i = u_i(x, y), \quad v_i = v_i(x, y), \quad w_i = \varepsilon_c^* z, \quad i = 1 \dots N+1, \quad (28)$$

where ε_c^* is a uniform transverse strain whose value may differ from the value ε^* that occurs in an undamaged laminate subject to the same applied stresses/displacements and the same temperature. The assumption (28) is equivalent to saying that stress transfer occurs only in the x and y directions. It follows from (5), (6) and (28) that, for $i = 1 \dots N+1$

$$\left. \begin{aligned} \varepsilon_{xz}^i &= \frac{1}{2} \left(\frac{\partial u_i}{\partial z} + \frac{\partial w_i}{\partial x} \right) \equiv 0, \\ \varepsilon_{yz}^i &= \frac{1}{2} \left(\frac{\partial v_i}{\partial z} + \frac{\partial w_i}{\partial y} \right) \equiv 0, \\ \varepsilon_{zz}^i &= \frac{\partial w_i}{\partial z} \equiv \varepsilon_c^*. \end{aligned} \right\} \quad (29)$$

Hence, from (9) and (10), for $i = 1 \dots N+1$

$$\sigma_{xz}^i \equiv \sigma_{yz}^i \equiv 0, \quad (30)$$

$$\sigma_{zz}^i = v_t^i \sigma_{xx}^i + v_A^i \frac{E_T^i}{E_A^i} \sigma_{yy}^i - E_T^i \alpha_T^i \Delta T + E_T^i \varepsilon_c^*. \quad (31)$$

Substituting (31) into (7) and (8) leads to

$$\varepsilon_{xx}^i = \frac{1}{\tilde{E}_t^i} \sigma_{xx}^i - \frac{\tilde{\nu}_a^i}{\tilde{E}_A^i} \sigma_{yy}^i + \alpha_t^i \Delta T - \nu_t^i \varepsilon_c^*, \quad (32)$$

$$\varepsilon_{yy}^i = -\frac{\tilde{\nu}_a^i}{\tilde{E}_A^i} \sigma_{xx}^i + \frac{1}{\tilde{E}_A^i} \sigma_{yy}^i + \alpha_A^i \Delta T - \nu_A^i \frac{E_T^i}{E_A^i} \varepsilon_c^*, \quad (33)$$

where

$$\left. \begin{aligned} \frac{1}{\tilde{E}_t^i} &= \frac{1}{E_t^i} - \frac{(\nu_t^i)^2}{E_T^i}, & \frac{\tilde{\nu}_a^i}{\tilde{E}_A^i} &= \frac{\nu_a^i + \nu_A^i \nu_t^i}{E_A^i}, \\ \frac{1}{\tilde{E}_A^i} &= \frac{1}{E_A^i} \left(1 - (\nu_A^i)^2 \frac{E_T^i}{E_A^i} \right), \\ \alpha_t^i &= \alpha_t^i + \nu_t^i \alpha_T^i, & \alpha_A^i &= \alpha_A^i + \nu_A^i \frac{E_T^i}{E_A^i} \alpha_T^i. \end{aligned} \right\} \quad (34)$$

Note that, from (17),

$$\tilde{E}_A^i = \hat{E}_A^i, \quad \alpha_A^i = \hat{\alpha}_A^i. \quad (35)$$

Thus, the following equations must be satisfied for all $i = 1 \dots N+1$

$$\varepsilon_{xx}^i \equiv \frac{\partial u_i}{\partial x} = \frac{1}{\tilde{E}_t^i} \sigma_{xx}^i - \frac{\tilde{\nu}_a^i}{\tilde{E}_A^i} \sigma_{yy}^i + \alpha_t^i \Delta T - \nu_t^i \varepsilon_c^*, \quad (36)$$

$$\varepsilon_{yy}^i \equiv \frac{\partial v_i}{\partial y} = -\frac{\tilde{\nu}_a^i}{\tilde{E}_A^i} \sigma_{xx}^i + \frac{1}{\tilde{E}_A^i} \sigma_{yy}^i + \alpha_A^i \Delta T - \nu_A^i \frac{E_T^i}{E_A^i} \varepsilon_c^*, \quad (37)$$

$$\varepsilon_{xy}^i \equiv \frac{1}{2} \left(\frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial x} \right) = \frac{\sigma_{xy}^i}{2\mu_a^i}, \quad (38)$$

$$\frac{\partial \sigma_{xx}^i}{\partial x} + \frac{\partial \sigma_{xy}^i}{\partial y} = 0, \quad \frac{\partial \sigma_{xy}^i}{\partial x} + \frac{\partial \sigma_{yy}^i}{\partial y} = 0. \quad (39)$$

5. INTERFACIAL AND BOUNDARY CONDITIONS

On imposing generalised plane strain conditions, the problem is two dimensional, in the x-y plane. Assuming solutions are symmetrical about both $x = 0$ and $y = 0$, we need only consider the positive quadrant $x \geq 0, y \geq 0$. The interfaces between the plies are defined to be at $x = x_i, i = 1 \dots N$, with $x_0 = 0$ and $x_{N+1} = h$. Note that $h_i = x_i - x_{i-1}, i = 1 \dots N+1$.

If perfect bonding is assumed between all the plies, then the relevant tractions and all displacements must be continuous throughout the laminate. Hence

$$\left. \begin{aligned} \sigma_{xx}^i &= \sigma_{xx}^{i+1}, & \sigma_{xy}^i &= \sigma_{xy}^{i+1} \\ u_i &= u_{i+1}, & v_i &= v_{i+1} \end{aligned} \right\} \text{ on } x = x_i, \quad i = 1 \dots N. \quad (40)$$

Symmetry about $x = 0$ implies that

$$\sigma_{xy}^1 \equiv u_1 \equiv 0 \quad \text{on } x = 0, \quad (41)$$

and, assuming that the outer surface of the plies is stress free,

$$\sigma_{xx}^{N+1} \equiv \sigma_{xy}^{N+1} \equiv 0 \quad \text{on } x = h. \quad (42)$$

Any undamaged layer surfaces on $y = \pm L$ are loaded by a given displacement, i.e.

$$v_i(x, \pm L) = \pm \epsilon_c L, \quad \sigma_{xy}^i(x, \pm L) = 0, \quad (43)$$

where ϵ_c is the effective axial strain applied to the cracked laminate. Any cracked surfaces on $y = \pm L$ will be stress free, i.e.

$$\sigma_{yy}^i(x, \pm L) = 0, \quad \sigma_{xy}^i(x, \pm L) = 0. \quad (44)$$

For the generalised plane strain model of a cracked laminate it is very easy to impose an applied transverse strain ϵ_c^* , although in practice it is more useful to specify a transverse applied stress σ_t . In this report it is assumed that σ_t is specified in all applications with the result that the transverse strain ϵ_c^* must be found as part of the solution. As seen from (31), the transverse stress for generalised plane strain conditions will depend upon the stress components σ_{xx} and σ_{yy} which will be non-uniform along the length of a cracked laminate. For generalised plane strain conditions a transverse applied stress condition is imposed using the following averaged boundary condition

$$\frac{1}{2hL} \sum_{i=1}^{N+1} \int_{-L}^L \int_{x_{i-1}}^{x_i} \sigma_{zz}^i dx dy = \sigma_t. \quad (45)$$

6. A MODEL FOR PREDICTING STRESS TRANSFER

Fundamental assumption

The problem of finding solutions to the equations (36-39) subject to the above boundary conditions can be solved using a recursive solution technique, analogous to that used by McCartney [14] for unidirectional composites. A fundamental single assumption is made that in each ply the stress component σ_{yy} is independent of x and is thus only a function of y . As a consequence it follows from (39)₂ that the shear stress σ_{xy} must take the form $xf_i(y) + g_i(y)$ in the i^{th} ply.

Stress field

Since the fundamental assumption leads to shear stresses σ_{xy} of the form $xf_i(y) + g_i(y)$, the following form for shear stress component σ_{xy}^i is valid for, $x_{i-1} \leq x \leq x_i$, $i = 1 \dots N+1$

$$\sigma_{xy}^i = \frac{1}{h_i} \left[C'_{i-1}(y) (x_i - x) + C'_i(y) (x - x_{i-1}) \right], \quad (46)$$

where the functions $C_i(y)$, $i = 0 \dots N+1$, are $N+2$ unknown functions to be determined. The prime is used to denote the derivative of the function C_i with respect to y ; this type of notation will also be used for higher order derivatives. From (46) it is easily shown that

$$\sigma_{xy}^i(x_i, y) \equiv \sigma_{xy}^{i+1}(x_i, y) \equiv C'_i(y), \quad i = 1 \dots N. \quad (47)$$

Thus, the shear stress component is automatically continuous at the interfaces, satisfying one of the conditions (40). Moreover, from (41) and (42) respectively

$$\left. \begin{aligned} \sigma_{xy}^1(x_0, y) &= C'_0(y) = 0, \\ \sigma_{xy}^{N+1}(x_{N+1}, y) &= C'_{N+1}(y) = 0. \end{aligned} \right\} \quad (48)$$

As the functions $C_i(y)$ characterise the perturbation from the solution given in section 3 for an undamaged laminate subject to the same applied stresses and displacements, it follows that $C_0(y) \equiv C_{N+1}(y) \equiv 0$. Thus, there are only N functions to be determined, namely $C_i(y)$, $i = 1 \dots N$. Note that the case where there is no stress transfer corresponds to values $C_i(y) \equiv 0$, $i = 0 \dots N+1$. Thus any form of stress transfer in the laminate is characterised by non-zero values of the functions $C_i(y)$.

The substitution of the expression (46) for σ_{xy}^i into the equilibrium equation (39)₁ followed by integration with respect to x leads to

$$\sigma_{xx}^i(x, y) = \frac{x - x_i}{2h_i} \left[(x - x_i) C_{i-1}''(y) - (x - x_i + 2h_i) C_i''(y) \right] + S_i(y), \quad i = 1 \dots N+1, \quad (49)$$

where the integration has been carried out so that $\sigma_{xx}^i(x_i, y) = S_i(y)$, where $S_i(y)$ is the through-the-thickness stress experienced by the i^{th} ply at the interface x_i . (Note from (42) that $S_{N+1}(y) \equiv 0$)

Similarly, after substituting the expression (46) for σ_{xy}^i into (39)₂ and integrating with respect to y it can be shown that

$$\sigma_{yy}^i(y) = \frac{1}{h_i} \left[C_{i-1}(y) - C_i(y) \right] + \sigma_i, \quad i = 1 \dots N+1, \quad (50)$$

where σ_i is the longitudinal stress experienced by the i^{th} ply of an undamaged laminate subject to the same applied stresses and displacements, an expression for which is derived later. Note that σ_{yy}^i has no x dependence, which is the fundamental assumption of the solution technique.

Displacement field

Substituting the expressions (49) and (50) for σ_{xx}^i and σ_{yy}^i into (36) leads to an expression for $\partial u_i / \partial x$ and on integrating with respect to x it can be shown that

$$u_i(x, y) = \frac{(x - x_i)^2}{6\tilde{E}_i^i h_i} \left[(x - x_i) C_{i-1}''(y) - (x - x_i + 3h_i) C_i''(y) \right] - \frac{x - x_i}{h_i} \frac{\nabla_a^i}{\tilde{E}_A^i} \left[C_{i-1}(y) - C_i(y) \right] + (x - x_i) \left[\frac{1}{\tilde{E}_t^i} S_i(y) + \varepsilon_i \right] + U_i(y), \quad i = 1 \dots N+1, \quad (51)$$

where the integration has been carried out so that $u_i(x_i, y) = U_i(y)$, where $U_i(y)$ is the through-the-thickness displacement at the interface x_i , and where ε_i is the through-the-thickness strain in the i^{th} ply of an undamaged laminate, subject to the same loading conditions, given by

$$\varepsilon_i = - \frac{\nabla_a^i}{\tilde{E}_A^i} \sigma_i + \alpha_i^i \Delta T - \nu_t^i \varepsilon_c^*. \quad (52)$$

Substituting (46) and (51) into (38) leads to an expression for $\partial v_i / \partial x$ and on integrating with respect to x it can be shown that

$$\begin{aligned}
v_i(x, y) = & \frac{x - x_i}{2h_i} \left[(x - x_i) \left(\frac{\bar{v}_a^i}{\bar{E}_A^i} - \frac{1}{\mu_a^i} \right) C'_{i-1}(y) \right. \\
& - \left. \left(\frac{\bar{v}_a^i}{\bar{E}_A^i} (x - x_i) - \frac{1}{\mu_a^i} (x - x_i + 2h_i) \right) C'_i(y) \right] \\
& - \frac{(x - x_i)^3}{24\bar{E}_t^i h_i} \left[(x - x_i) C'''_{i-1}(y) - (x - x_i + 4h_i) C'''_i(y) \right] \\
& - \frac{(x - x_i)^2}{2\bar{E}_t^i} S'_i(y) - (x - x_i) U'_i(y) + V_i(y), \quad i = 1 \dots N+1,
\end{aligned} \tag{53}$$

where the integration has been carried out such that $v_i(x_i, y) = V_i(y)$, where $V_i(y)$ is the longitudinal displacement at the interfaces $x = x_i$, $i = 1 \dots N+1$.

The functions S_i , U_i , and V_i are clearly defined as the values of $\sigma_{xx}(x, y)$, $u_i(x, y)$, and $v_i(x, y)$ respectively at the interfaces $x = x_i$, for $i = 1 \dots N$ and on the external surface $x = x_{N+1}$. This definition can be extended without loss of generality so that S_0 , U_0 and V_0 are the values of these functions at x_0 . From (41) it is clear that $U_0 = 0$. The values of S_0 and V_0 are obtained from (49) and (53) on setting $i = 1$ and letting $x = x_0 = 0$.

Recurrence relations

Subjecting (49), (51) and (53) to the relevant interfacial continuity conditions (40) leads to the following three recurrence relations generating values of S_i , U_i and V_i

$$S_{i-1} = S_i + \frac{h_i}{2} [C''_{i-1} + C''_i], \quad S_{N+1} = 0, \tag{54}$$

for $i = N+1 \dots 1$, and

$$U_i = U_{i-1} + \frac{h_i^2}{6\bar{E}_t^i} [C''_{i-1} + 2C''_i] - \frac{\bar{v}_a^i}{\bar{E}_A^i} [C_{i-1} - C_i] + h_i \left[\frac{1}{\bar{E}_t^i} S_i + \bar{\epsilon}_i \right], \quad U_0 = 0, \tag{55}$$

$$\begin{aligned}
V_i = & V_{i-1} + \frac{h_i^3}{24\bar{E}_t^i} [C'''_{i-1} + 3C'''_i] - \frac{h_i}{2} \left[\left(\frac{\bar{v}_a^i}{\bar{E}_A^i} - \frac{1}{\mu_a^i} \right) C'_{i-1} - \left(\frac{\bar{v}_a^i}{\bar{E}_A^i} + \frac{1}{\mu_a^i} \right) C'_i \right] \\
& + \frac{h_i^2}{2\bar{E}_t^i} S'_i - h_i U'_i, \quad i = 1 \dots N+1.
\end{aligned} \tag{56}$$

From (56) the functions V_i , $i = 2 \dots N+1$, can be obtained in terms of V_1 .

7. REDUCTION OF PROBLEM TO ORDINARY DIFFERENTIAL EQUATIONS

The solution given by equations (46) to (53) satisfies exactly the equations (36), (38) and (39), but it cannot satisfy (37) exactly. To resolve this problem, (37) is replaced by its average through the thickness of the laminate, following the procedure used by McCartney [10] for the case of two ply models.

Averaging

If $f_i(x, y)$ is any variable associated with the i^{th} ply, then its average is defined by

$$\bar{f}_i(y) = \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f_i(x, y) dx . \quad (57)$$

Averaging v_i using (53) and (57) leads to, for $i = 1 \dots N+1$

$$\begin{aligned} \bar{v}_i(y) = & \frac{h_i}{6} \left[\left(\frac{\bar{v}_a^i}{\bar{E}_A^i} - \frac{1}{\mu_a^i} \right) C_{i-1}'(y) - \left(\frac{\bar{v}_a^i}{\bar{E}_A^i} + \frac{2}{\mu_a^i} \right) C_i'(y) \right] \\ & - \frac{h_i^3}{120\bar{E}_t^i} (C_{i-1}'''(y) + 4C_i'''(y)) - \frac{h_i^2}{6\bar{E}_t^i} S_i'(y) + \frac{h_i}{2} U_i'(y) + V_i(y) , \end{aligned} \quad (58)$$

and averaging σ_{xx}^i and σ_{yy}^i using (49), (50) and (57) leads to, for $i = 1 \dots N+1$

$$\bar{\sigma}_{xx}^i(y) = \frac{h_i}{6} [C_{i-1}''(y) + 2C_i''(y)] + S_i(y) , \quad (59)$$

$$\bar{\sigma}_{yy}^i(y) = \frac{1}{h_i} [C_{i-1}(y) - C_i(y)] + \sigma_i . \quad (60)$$

Using (58-60) the average of (37) may be expressed in the following form, for $i = 1 \dots N+1$

$$\begin{aligned} V_i'(y) = & \frac{h_i^3}{120\bar{E}_t^i} [C_{i-1}'''' + 4C_i'''] - \frac{h_i}{6} \left[\left(2\frac{\bar{v}_a^i}{\bar{E}_A^i} - \frac{1}{\mu_a^i} \right) C_{i-1}'' + \left(\frac{\bar{v}_a^i}{\bar{E}_A^i} - \frac{2}{\mu_a^i} \right) C_i'' \right] \\ & + \frac{1}{h_i\bar{E}_A^i} [C_{i-1} - C_i] + \frac{h_i^2}{6\bar{E}_t^i} S_i'' - \frac{h_i}{2} U_i'' - \frac{\bar{v}_a^i}{\bar{E}_A^i} S_i + \epsilon , \end{aligned} \quad (61)$$

where the parameter ϵ is the axial strain in an undamaged laminate, subject to the same loading conditions, given by

$$\varepsilon = \frac{1}{\tilde{E}_A^i} \sigma_i - \nu_A^i \frac{E_T^i}{\tilde{E}_A^i} \varepsilon_c^* + \alpha_A^i \Delta T . \quad (62)$$

On integrating (61) and substituting into (58), it can be shown that the expression for $\bar{v}_i(y)$ reduces to the simpler form

$$\bar{v}_i(y) = -\frac{h_i \nu_a^i}{6\tilde{E}_A^i} [C_{i-1}'(y) + 2C_i'(y)] + \frac{1}{h_i \tilde{E}_A^i} [C_{i-1}^*(y) - C_i^*(y)] - \frac{\nu_a^i}{\tilde{E}_A^i} S_i^*(y) + \varepsilon y , \quad (63)$$

where

$$C_i^*(y) = \int_0^y C_i(s) ds , \quad (64)$$

and where

$$S_i^*(y) = \int_0^y S_i(s) ds . \quad (65)$$

Simultaneous differential equations

The recurrence relation (54) can be used to generate values of S_i with the result that

$$S_i = \sum_{j=i+1}^{N+1} \frac{h_j}{2} [C_{j-1}'' + C_j''] , \quad i = 0 \dots N , \quad S_{N+1} \equiv 0 , \quad (66)$$

and then S_i can be substituted into (55) to generate the following expressions for U_i

$$\begin{aligned} U_i = \sum_{j=1}^i \left[\frac{h_j^2}{6\tilde{E}_t^j} (C_{j-1}'' + 2C_j'') - \frac{\nu_a^j}{\tilde{E}_A^j} (C_{j-1} - C_j) \right] \\ + \sum_{j=1}^i \frac{h_j}{\tilde{E}_t^j} \left[\sum_{k=j+1}^{N+1} \frac{h_k}{2} (C_{k-1}'' + C_k'') \right] + \sum_{j=1}^i h_j \xi_j , \quad i = 1 \dots N , \quad U_0 \equiv 0 . \end{aligned} \quad (67)$$

Differentiating (56) with respect to y leads to

$$V'_i = V'_{i-1} + \frac{h_i^3}{24\tilde{E}_i} [C_{i-1}^{''''} + 3C_i^{''''}] - \frac{h_i}{2} \left[\left(\frac{\tilde{v}_a^i}{\tilde{E}_A^i} - \frac{1}{\mu_a^i} \right) C_{i-1}'' - \left(\frac{\tilde{v}_a^i}{\tilde{E}_A^i} + \frac{1}{\mu_a^i} \right) C_i'' \right] + \frac{h_i^2}{2\tilde{E}_i} S_i'' - h_i U_i'', \quad i = 1 \dots N+1. \quad (68)$$

Now define $\Delta V'_i(y)$ as

$$\Delta V'_i(y) \equiv V'_i(y) - V'_1(y), \quad i = 1 \dots N+1. \quad (69)$$

Clearly, $\Delta V'_1(y) \equiv 0$ and on subtracting $V'_1(y)$ from both sides of (68)

$$\Delta V'_i = \Delta V'_{i-1} + \frac{h_i^3}{24\tilde{E}_i} [C_{i-1}^{''''} + 3C_i^{''''}] - \frac{h_i}{2} \left[\left(\frac{\tilde{v}_a^i}{\tilde{E}_A^i} - \frac{1}{\mu_a^i} \right) C_{i-1}'' - \left(\frac{\tilde{v}_a^i}{\tilde{E}_A^i} + \frac{1}{\mu_a^i} \right) C_i'' \right] + \frac{h_i^2}{2\tilde{E}_i} S_i'' - h_i U_i'', \quad i = 2 \dots N+1, \quad (70)$$

which is a recurrence relation generating the following values of $\Delta V'_i(y)$

$$\Delta V'_i = \sum_{j=2}^i \left[\frac{h_j^3}{24\tilde{E}_j} (C_{j-1}^{''''} + 3C_j^{''''}) - \frac{h_j}{2} \left\{ \left(\frac{\tilde{v}_a^j}{\tilde{E}_A^j} - \frac{1}{\mu_a^j} \right) C_{j-1}'' - \left(\frac{\tilde{v}_a^j}{\tilde{E}_A^j} + \frac{1}{\mu_a^j} \right) C_j'' \right\} + \frac{h_j^2}{2\tilde{E}_j} S_j'' - h_j U_j'' \right], \quad \text{for } i = 2 \dots N+1, \quad \Delta V'_1 \equiv 0. \quad (71)$$

The equation (61) must also be satisfied which leads to explicit expressions for the functions $V'_i(y)$. Using (61) and (71) to generate values of V'_i and $\Delta V'_i$ respectively, define

$$\phi_j(y) = V'_{j+1}(y) - V'_1(y) - \Delta V'_{j+1}(y), \quad j = 1 \dots N, \quad (72)$$

which must be identically zero if (69) is to be satisfied. On using (61), (66) and (67) to calculate $V'_j(y)$ and (66), (67) and (71) to calculate $\Delta V'_j(y)$, it can be shown that the expression for $\phi_j(y)$ only involves the unknown functions $C_i(y)$, and is of the following homogeneous form

$$\phi_j(y) = \sum_{i=1}^N F_{ij} C_i''''(y) + \sum_{i=1}^N G_{ij} C_i''(y) + \sum_{i=1}^N H_{ij} C_i(y) \equiv 0, \quad j = 1 \dots N, \quad (73)$$

which defines N simultaneous differential equations for the N unknowns $C_i(y)$, $i = 1 \dots N$. The coefficients F_{ij} , G_{ij} and H_{ij} are constants which are extremely complicated expressions generated by recurrence relations. These coefficients need to be calculated before the solution of (73) can be determined.

Calculation of the coefficients

It is convenient to denote the j^{th} equation of the set (73) by $\phi_j(C, C'', C''')$, where $C \equiv \{C_i, i = 1 \dots N\}$, having second and fourth derivatives denoted by C'' and C'''' respectively. The numerical values of the coefficients F_{ij} , G_{ij} and H_{ij} , $i, j = 1 \dots N$, are then calculated simply by obtaining values of ϕ_j , $j = 1 \dots N$ using suitable sets of values for the functions C , C'' and C'''' . Thus

$$\begin{aligned} F_{ij} &\equiv \phi_j(0, 0, I_i), \\ G_{ij} &\equiv \phi_j(0, I_i, 0), \quad i = 1 \dots N, \quad j = 1 \dots N, \\ H_{ij} &\equiv \phi_j(I_i, 0, 0), \end{aligned} \quad (74)$$

with $I_i \equiv \{\delta_{ik}, k = 1 \dots N\}$ where δ_{ik} is the Kronecker delta, and with 0 denoting a corresponding set of zero values. Using values of the coefficients resulting from such calculations the functions C_i , $i = 1 \dots N$, can then be determined from (73) using numerical methods [15] as soon as appropriate additional boundary conditions are imposed. This method of calculating coefficients is analogous to the approach used by McCartney [14].

8. APPLICATION OF BOUNDARY CONDITIONS

Three independent parameters are needed to describe the loading and temperature applied to the cracked laminate. They are selected to be axial strain ϵ for an undamaged laminate subject to the same loading conditions and temperature as the cracked laminate, the transverse strain ϵ_c^* and the temperature difference ΔT . Other parameters that must be found, as part of the solution, are the corresponding effective axial applied stress σ , the effective axial strain of a cracked laminate ϵ_c and the effective transverse stress σ_t acting on the cracked laminate. As the cracks in the plies are assumed to be co-planar in a set of parallel equally spaced planes, symmetry arguments together with the fact that cracked plies are stress-free on $y = L$, can be used to conclude that $\sigma_{xy}^i \equiv 0$ for all values of x on both $y = 0$ and $y = \pm L$. It is thus sufficient to consider only the region $0 \leq y \leq L$, and from (46) it is clear that the following $2N$ of the required $4N$ boundary conditions are obtained, namely

$$C_i'(0) = 0, \quad C_i'(L) = 0, \quad i = 1 \dots N. \quad (75)$$

A further N boundary conditions to be imposed ensure the symmetry of the functions C_i about $y = 0$. The condition needed is obtained from the integrated form of the differential equations (73), namely

$$\sum_{i=1}^N F_{ij} C_i'''(y) + \sum_{i=1}^N G_{ij} C_i'(y) + \sum_{i=1}^N H_{ij} C_i^*(y) \equiv 0, \quad j = 1 \dots N. \quad (76)$$

On using (64) and (75)₁ it is clear that the relations (76) are always satisfied at $y = 0$ only if the following conditions are satisfied

$$C_i'''(0) = 0, \quad i = 1 \dots N. \quad (77)$$

To obtain some of the remaining N required boundary conditions, use is made of the stress boundary condition (44)₁ applied on $y = L$ in those plies that are cracked. The equation (50) then leads to the conditions

$$C_{i-1}(L) - C_i(L) + h_i \sigma_i = 0, \quad (\text{cracked plies only}), \quad (78)$$

where from (62) the parameters σ_i are given immediately, in terms of ϵ , ϵ_c^* and ΔT , by

$$\sigma_i = \tilde{E}_A^i \left[\epsilon + v_A^i \frac{E_T^i}{E_A^i} \epsilon_c^* - \alpha_A^i \Delta T \right]. \quad (79)$$

As $C_0(y) \equiv C_{N+1}(y) \equiv 0$ it follows from (50) that the relation (14)₁ remains valid for cracked laminates. Thus the axial stress σ applied to the cracked laminate may be calculated using (14)₁. The result of this calculation may be written in the following form corresponding to (19)

$$\sigma = A\epsilon + B\epsilon_c^* - P\Delta T, \quad (80)$$

where it can be shown using (35) that the parameters A , B and P are given by (21) and (22). Thus the effective axial applied stress σ is readily calculated from the given axial and transverse strains, and the temperature difference.

The transverse stress boundary condition (45), for generalised plane strain conditions, has not so far been imposed. On substituting (49) and (50) in (31), it can be shown that

$$\begin{aligned} \sigma_{zz}^i(x, y) = v_t^i \left[\frac{x - x_i}{2h_i} \left\{ (x - x_i) C_{i-1}''(y) - (x - x_i + 2h_i) C_i''(y) \right\} + S_i(y) \right] \\ + v_A^i \frac{E_T^i}{E_A^i} \frac{C_{i-1}(y) - C_i(y)}{h_i} + \sigma_i^*, \quad i = 1 \dots N+1, \end{aligned} \quad (81)$$

where

$$\sigma_i^* = v_A^i \frac{E_T^i}{E_A^i} \sigma_i - E_T^i \alpha_T^i \Delta T + E_T^i \epsilon_c^*. \quad (82)$$

Before integrating (81) along the length of the laminate it is useful to derive from (65) and (66) the following expression

$$S_i^*(y) = \sum_{j=i+1}^{N+1} \frac{h_j}{2} [C'_{j-1}(y) + C'_j(y)], \quad i = 0 \dots N, \quad \text{where } S_{N+1}^*(y) = 0. \quad (83)$$

It follows on using (48), (65), (75), (81) and (83), together with the symmetry of the function C_i about $y = 0$, that

$$\sigma_t^i = \frac{1}{hL} \int_0^L \int_{x_{i-1}}^{x_i} \sigma_{zz}(x, y) dx dy = v_A^i \frac{E_T^i}{E_A^i} \frac{C_{i-1}^*(L) - C_i^*(L)}{h_i L} + \sigma_i^*, \quad i = 1 \dots N+1, \quad (84)$$

where σ_t^i is the average transverse stress acting on the i^{th} ply. On multiplying by h_i and summing over i using the definition (14)₂ for σ^* it follows that the average transverse stress σ_v defined by (45) and acting on the laminate as a whole, is given by

$$\sigma_t = \frac{1}{h} \sum_{i=1}^{N+1} h_i \sigma_t^i = \frac{1}{hL} \sum_{i=1}^{N+1} v_A^i \frac{E_T^i}{E_A^i} [C_{i-1}^*(L) - C_i^*(L)] + \sigma^*, \quad (85)$$

where on substituting (79) in (82), the following expression for σ^* is obtained corresponding to (20)

$$\sigma^* = \frac{1}{h} \sum_{i=1}^{N+1} h_i \sigma_i^* = B\varepsilon + C\varepsilon_c^* - Q\Delta T. \quad (86)$$

It can be shown, on using (17), (18) and (35), that

$$B = \frac{1}{h} \sum_{i=1}^{N+1} h_i v_A^i \frac{E_T^i}{E_A^i} \tilde{E}_A^i = \frac{1}{h} \sum_{i=1}^{N+1} h_i v_A^i \hat{E}_T^i, \quad (87)$$

$$C = \frac{1}{h} \sum_{i=1}^{N+1} h_i \left[E_T^i + \left(v_A^i E_T^i / E_A^i \right)^2 \tilde{E}_A^i \right] = \frac{1}{h} \sum_{i=1}^{N+1} h_i \hat{E}_T^i, \quad (88)$$

$$Q = \frac{1}{h} \sum_{i=1}^{N+1} h_i \left[E_T^i \alpha_T^i + v_A^i \frac{E_T^i}{E_A^i} \tilde{E}_A^i \alpha_A^i \right] = \frac{1}{h} \sum_{i=1}^{N+1} h_i \hat{E}_T^i \alpha_T^i. \quad (89)$$

These results are consistent with (21) and (22). From (85) it is clear that the effective transverse stress σ_t is readily calculated, and that $\sigma_t = \sigma^*$ when the laminate is undamaged. It is worth noting that on using (25) and (26) it can be shown that

$$A = \frac{E_A}{1 - v_A^2 \frac{E_T}{E_A}}, \quad B = \frac{v_A E_T}{1 - v_A^2 \frac{E_T}{E_A}}, \quad C = \frac{E_T}{1 - v_A^2 \frac{E_T}{E_A}}, \quad (90)$$

$$P = \frac{E_A \alpha_A + v_A E_T \alpha_T}{1 - v_A^2 \frac{E_T}{E_A}}, \quad Q = \frac{E_T [\alpha_T + v_A \alpha_A]}{1 - v_A^2 \frac{E_T}{E_A}}. \quad (91)$$

Use is now made of the boundary condition (43)₁ relating to uncracked plies. As this condition cannot be satisfied exactly by the relation (53) because of the x dependence, the following less restrictive equivalent average condition is imposed on $y = L$, namely

$$\bar{v}_i(L) = L \epsilon_c, \quad (\text{only for uncracked plies}), \quad (92)$$

where ϵ_c is the effective axial strain applied to the cracked laminate and where $\bar{v}_i(y)$ is defined by (63). From (48), (63), (75)₂ and (83), the condition (92) for an uncracked ply at $y = L$ may be written in the form of an expression for the axial strain ϵ_c of the cracked laminate as follows

$$\epsilon_c = \epsilon - \frac{1}{h_i \bar{E}_A^i L} [C_i^*(L) - C_{i-1}^*(L)], \quad (\text{only for uncracked plies}), \quad (93)$$

so that $\epsilon_c = \epsilon$ when the laminate is undamaged.

For the special case when (in region $x > 0$) there is just one uncracked ply and N cracked plies, the boundary conditions that lead to a unique solution of the differential equations (73) are given by the $4N$ relations (75), (77) and (78). For this case the relation (93) is used only to calculate the effective axial strain ϵ_c of the laminate. For cases where there are $m + 1$ uncracked plies where $m > 0$, there are $N - m$ cracked plies leading to $N - m$ boundary conditions of the form (78), and m boundary conditions of the following form obtained by eliminating the parameter ϵ_c using (93)

$$\frac{1}{h_i \bar{E}_A^i} [C_i^*(L) - C_{i-1}^*(L)] - \frac{1}{h_j \bar{E}_A^j} [C_j^*(L) - C_{j-1}^*(L)] = 0, \quad (94)$$

(only for uncracked plies where $i \neq j$)

where the j^{th} crack is selected to be any one of the uncracked plies.

It is convenient to combine boundary conditions of the type (78) and (94) into a set of N conditions having the same form. To achieve this it is necessary to introduce a set of parameters λ_i , $i = 1 \dots N+1$, such that the value of λ_i specifies whether or not the i^{th} ply is cracked. The convention used is to select $\lambda_i = 1$ if the i^{th} ply is cracked and the value $\lambda_i = 0$ if it is uncracked. The conditions (78) and (94) may then be combined into the following set of N boundary conditions

$$\begin{aligned} p_i [C_i(L) - C_{i-1}(L)] + q_i [C_i^*(L) - C_{i-1}^*(L)] \\ - (1 - \lambda_i) q_i [C_j^*(L) - C_{j-1}^*(L)] + r_i = 0 \quad \text{for } i \neq j, \end{aligned} \quad (95)$$

where

$$p_i = \lambda_i, \quad q_i = \frac{1 - \lambda_i}{h_i \tilde{E}_A^i}, \quad r_i = -\lambda_i h_i \sigma_i, \quad i = 1 \dots N+1. \quad (96)$$

Thus the relations specified by (75), (77) and (95) provide the $4N$ conditions required to obtain a unique solution of the differential equations (73) when there are two or more uncracked plies in the region $x > 0$.

9. CALCULATION OF ELASTIC CONSTANTS

By making use of the analysis presented in sections 6-8, together with the numerical method [15] of solving the simultaneous ordinary differential equations (73) subject to the $4N$ boundary conditions that assume that the axial applied strain ε in an undamaged laminate, the transverse applied strain ε_c^* in a cracked laminate and the temperature difference ΔT are specified, it is possible to calculate the effective in-plane thermoelastic constants using the following method. The effective stress-strain-temperature relations for the cracked laminate may be written

$$\varepsilon_c = \frac{\sigma}{E_A^c} - \frac{\nu_A^c}{E_A^c} \sigma_t + \alpha_A^c \Delta T, \quad (97)$$

$$\varepsilon_c^* = -\frac{\nu_A^c}{E_A^c} \sigma + \frac{\sigma_t}{E_T^c} + \alpha_T^c \Delta T, \quad (98)$$

where E_A^c , E_T^c , ν_A^c , α_A^c and α_T^c denote the effective thermoelastic constants of the cracked laminate. It follows from (80) that the effective axial applied stress σ can easily be calculated without needing to have access to the solution of the differential equations that satisfy the prescribed boundary conditions. Thus for a cracked laminate the parameters σ , ε_c^* and ΔT can be regarded as being specified independent parameters. On using (98) the effective transverse stress σ_t applied to the cracked laminate may then be written

$$\sigma_t = E_T^c \left[\varepsilon_c^* + \frac{v_A^c}{E_A^c} \sigma - \alpha_T^c \Delta T \right] \equiv f(\varepsilon_c^*, \sigma, \Delta T) . \quad (99)$$

On substituting (99) in (97) the effective axial strain in the cracked laminate is given by

$$\varepsilon_c = -v_A^c \frac{E_T^c}{E_A^c} \varepsilon_c^* + \frac{1}{E_A^c} \left[1 - (v_A^c)^2 \frac{E_T^c}{E_A^c} \right] \sigma + \left[\alpha_A^c + v_A^c \frac{E_T^c}{E_A^c} \alpha_T^c \right] \Delta T \equiv g(\varepsilon_c^*, \sigma, \Delta T) . \quad (100)$$

The thermoelastic constants may then be determined from the relations

$$E_T^c = f(1, 0, 0) , \quad (101)$$

$$\frac{v_A^c}{E_A^c} = \frac{f(0, 1, 0)}{E_T^c} = - \frac{g(1, 0, 0)}{E_T^c} , \quad (102)$$

$$\alpha_T^c = - \frac{f(0, 0, 1)}{E_T^c} , \quad (103)$$

$$E_A^c = \frac{1}{g(0, 1, 0) + (v_A^c / E_A^c)^2 E_T^c} , \quad (104)$$

$$\alpha_A^c = g(0, 0, 1) - \frac{v_A^c}{E_A^c} E_T^c \alpha_T^c . \quad (105)$$

Thus the effective thermoelastic constants of the cracked laminate are readily calculated by solving the cracked laminate problem for three independent cases specified by the following values of the parameters ε_c^* , σ and ΔT

$$\begin{aligned}
\varepsilon_c^* &= 1, & \sigma &= 0, & \Delta T &= 0, \\
\varepsilon_c^* &= 0, & \sigma &= 1, & \Delta T &= 0, \\
\varepsilon_c^* &= 0, & \sigma &= 0, & \Delta T &= 1.
\end{aligned} \tag{106}$$

For each of these cases the axial strain ε is first calculated using (80). On using ε , ε_c^* , and ΔT as input parameters, the function $f(\varepsilon_c^*, \sigma, \Delta T)$ is calculated using the expression (85), and the function $g(\varepsilon_c^*, \sigma, \Delta T)$ is calculated using (93).

In many practical applications, it is desirable to specify the effective axial and transverse applied stresses σ and σ_t respectively, and the temperature difference ΔT . Solutions for this case are readily obtained once the corresponding values of ε and ε_c^* are found. The transverse strain for the cracked laminate ε_c^* is obtained directly from (98) using the calculated values for the thermoelastic constants of the cracked laminate. The axial strain ε for an uncracked laminate subject to the same loading conditions follows from (80) and may be expressed on using (90) and (91) in the form

$$\varepsilon = \frac{1}{E_A} \left[1 - \nu_A^2 \frac{E_T}{E_A} \right] \sigma - \nu_A \frac{E_T}{E_A} \varepsilon_c^* + \left[\alpha_A + \nu_A \frac{E_T}{E_A} \alpha_T \right] \Delta T. \tag{107}$$

Thus by making use of the derived values for the thermoelastic constants of the cracked laminate, detailed stress and displacement distributions can be obtained for any biaxial applied stresses σ and σ_t by repeating the solution of the differential equations (73) subject to the prescribed boundary conditions using the values of ε and ε_c^* specified respectively by (107) and (98).

10. CONCLUSION

The analysis described in this paper shows how the stress and displacement fields may be calculated at all points in a multiple-ply cross-ply laminate, taking account of the presence of a uniform distribution of transverse cracks in the 90° plies, and of thermal residual stresses. The technique reduces the laminate stress transfer problem to the solution of a set of fourth order ordinary differential equations that are solved by numerical techniques. It has also been shown how such solutions can be used to calculate the thermo-elastic constants of a cracked laminate in terms of the crack separation.

It is beyond the scope of this report to present results obtained using the analysis. Preliminary results for a glass reinforced plastic laminate have been presented in reference [16]. Many more applications have been attempted including predictions for carbon fibre reinforced plastic laminates. To date the technique has proved itself to be very accurate, if ply refinement methods are used, and also robust. A distinct advantage of the method is that it can be used to check upon accuracy of solutions by investigating how well boundary conditions are satisfied. The results of applications of the technique are to be published.

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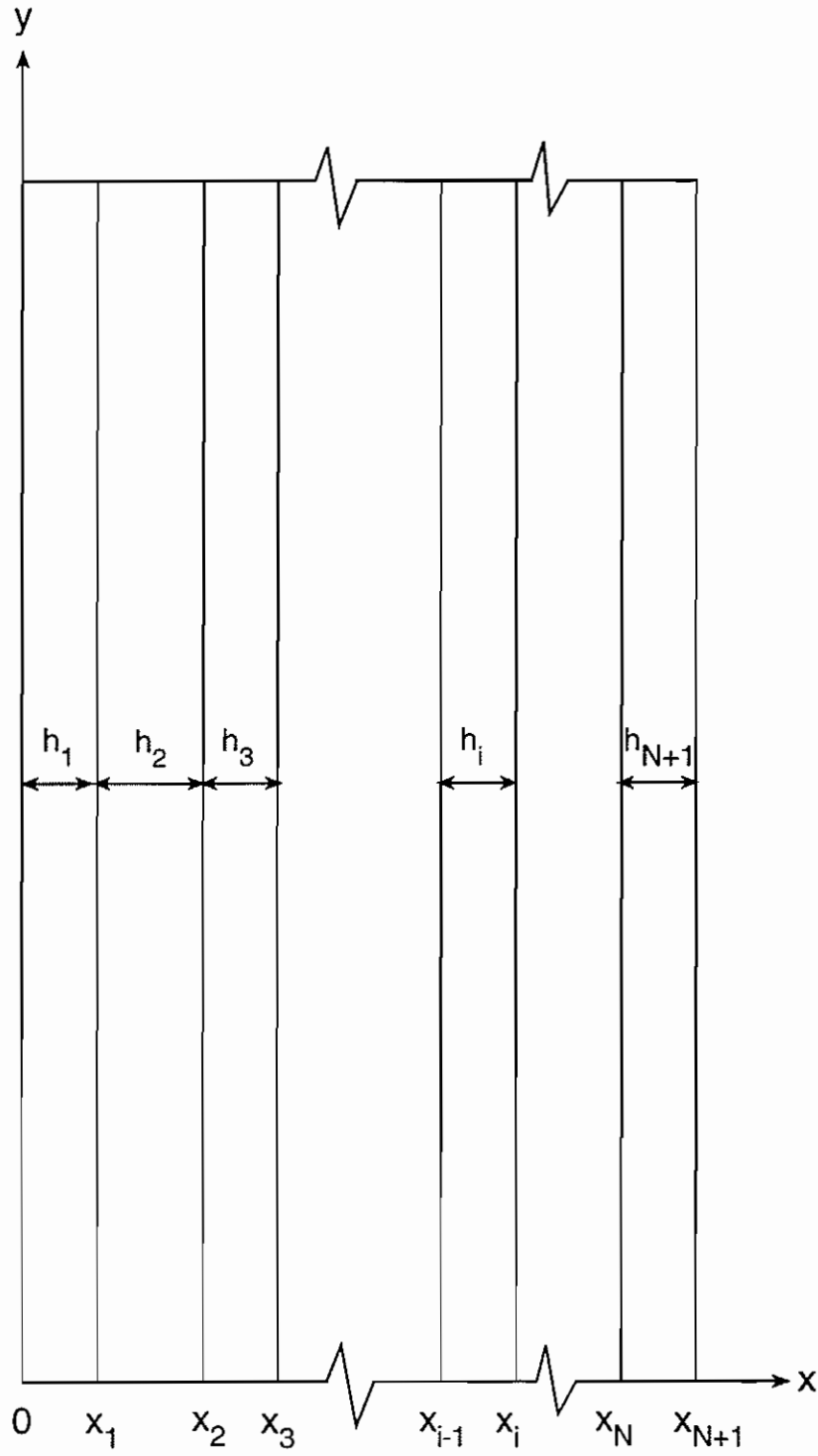


FIG:1 Schematic edge view of the right hand half of a multiple-ply cross-ply laminate indicating coordinate axes and geometrical interpretation of the ply parameters h_i and x_i , $i = 1 \dots N+1$.

