

How to deduce the far field of an antenna from planar near-field measurements

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ABSTRACT

This report sets out the mathematical basis for transforming measurements of a vector near field over a plane to the corresponding far field. The theory outlined here is the basis for computer programs used in the radio-frequency and microwave antenna group at NPL to deduce the far field of an antenna from planar near-field measurements.

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FREQUENTLY USED SYMBOLS

E	electric field
H	magnetic field
ω	angular frequency
c	speed of light, =299 792 458 m/s <i>in vacuo</i>
k	ω/c , the free-space wavenumber

1 SCOPE OF THIS REPORT

There is a growing demand for precise measurements of the gain and radiation pattern of microwave antennas. Some are high-gain antennas and their Rayleigh distance is so great that it is inconvenient to measure the far field directly or by extrapolation techniques. Instead, the near field is scanned and the far field is determined from the near-field scan (see *e.g.* Yaghjian 1986). Common scanning geometries are planar (Baird et al. 1988, Wang 1988), spherical (Hansen and Jensen 1988) and cylindrical (Yaghjian 1986, Bucci 1988). At NPL we have a planar near-field scanner measurement facility; this report describes the mathematical basis of the transformation we use to deduce the far field from planar near-field data.

The problem has been cast in terms of the electric field E throughout. E is a vector, and each of its three spatial components is represented by a complex number containing both amplitude and phase information. The scalar transformation problem is a subset of the vector problem, and the method set out in this report may be used to transform scalar fields. For example, there is an analogous transformation problem in acoustics, but with rather different boundary conditions.

Under certain conditions, there is a direct mathematical expression for the far field in terms of an integral of the near field over the measurement plane. This relation and these conditions are derived in chapter 2 of this report.

In fact, we never start from a complete and accurate knowledge of the near field. In chapter 3, the exact relations derived in chapter 2 are used to build a mathematical transformation algorithm for predicting the far-field of an antenna, given real, imperfect measurements of the near field.

2 THEORY

In free space, Maxwell's curl equations (see e.g. Feynmann 1964) combine to give for the electric field E the equation

$$\frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = \nabla^2 E - \nabla(\nabla \cdot E)$$

which, with the free-space divergence condition $\text{div}(E)=0$, reduces to

$$\frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = \nabla^2 E$$

Continuous-wave (monochromatic) solutions to (2.1) exist and we are free to choose between two conventions for writing down their time-dependence. In this report, the physicists' convention

$$E \sim e^{-i\omega t}$$

is used, where ω is the angular frequency of radiation. (The engineers' convention $E \sim e^{+i\omega t}$ would be equally acceptable, but it is important to use the same definition throughout.) This choice implies that phase at a fixed point in space decreases as time progresses; at a fixed moment in time, phase increases with increasing distance away from a source of radiation.

Equation (2.1) now reduces to the vector Helmholtz equation for E :

$$\nabla^2 E + k_0^2 E = 0$$

where $k_0 = \omega/c$, c being the speed of light. The cartesian components of E each obey three separate scalar Helmholtz equations, linked by a common value of k_0 .

$$\begin{aligned} \nabla^2 E_x + k_0^2 E_x &= 0 \\ \nabla^2 E_y + k_0^2 E_y &= 0 \\ \nabla^2 E_z + k_0^2 E_z &= 0 \end{aligned} \tag{2.3}$$

Separability of the vector Helmholtz equation is unique to the cartesian representation of the vector E . It means that, if E_x , say, is zero throughout the near field, it will be zero everywhere, because there is no mixing between the three equations (2.3). This makes cartesian coordinates a natural basis for our transformation. It also influences our choice of a set of basis vectors in the far field, as described in section 2.6.

2.1 The scalar transformation: Rayleigh-Sommerfeld diffraction theory

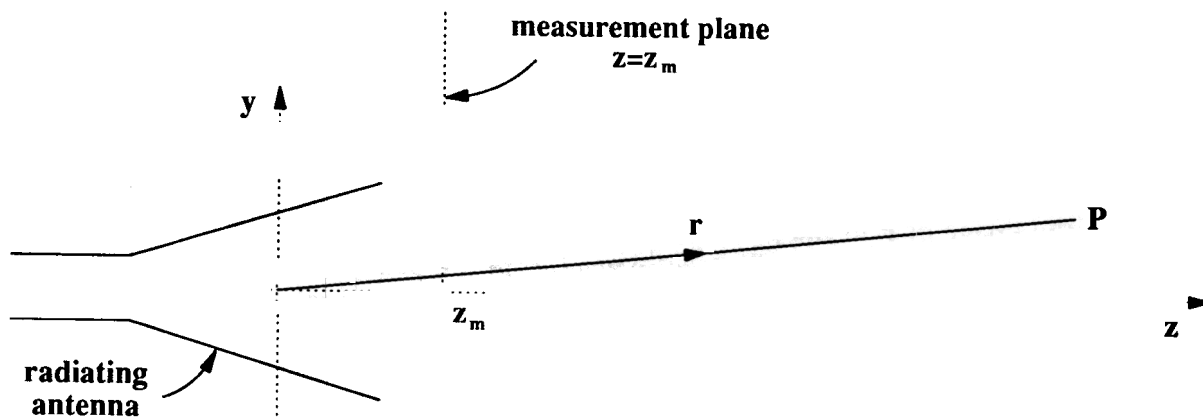


Figure 2.1

Let e stand for any one of the three cartesian components E_x, E_y, E_z of E . Suppose that we want to determine the field e at a point P , with position vector $r=(x,y,z)$ in the half-space $z \geq z_m$, due to an antenna located entirely in the half-space $z < z_m$ (figure 2.1). The planar near-field to far-field transformation problem may now be stated as follows:

Given the function $e(x,y,z_m)$ all over the infinite plane $z=z_m$, find an expression for e in $z \geq z_m$ as $r \rightarrow \infty$, where $r=(x,y,z)$.

The Rayleigh-Sommerfeld formulation of Kirchhoff diffraction theory (Sommerfeld 1949, Goodman 1968, Born and Wolf 1980, Nye unpublished 1988) addresses exactly this problem; the following is a brief summary of Sommerfeld's method.

Let G be a second function satisfying the same Helmholtz equation as e ; so we have

$$\begin{aligned}\nabla^2 e + k_0^2 e &= 0 \\ \nabla^2 G + k_0^2 G &= 0\end{aligned}$$

and we can use Green's theorem, with the half-space $z \geq z_m$ as the interior region (Sommerfeld 1949, p.197) to show that

$$e(x,y,z \geq z_m) = \left(\frac{1}{4\pi} \right) \iint_{z_0=z_m} \left(G \frac{\partial e(x_0,y_0,z_0)}{\partial z_0} - e(x_0,y_0,z_0) \frac{\partial G}{\partial z_0} \right) dx_0 dy_0 \quad (2.4)$$

where the integral is evaluated all over the infinite plane $z=z_m$. For G , we use the Rayleigh-Sommerfeld Green's function

$$G = \frac{e^{ik_0 r_0}}{r_0} - \frac{e^{ik_0 r'_0}}{r'_0}$$

where r_0 is the length of the vector $\mathbf{r}_0$ between the points $\{P_0=(x_0, y_0, z_0)\}$ and $P=(x, y, z)\}$ and where r'_0 is the length of the vector $\mathbf{r}'_0$ between $\{P_0=(x_0, y_0, z_0)\}$ and $P'=(x, y, 2z_0-z)\}$; this last point P' is the image of P in the plane $z=z_0$. The definition of G is illustrated in figure (2.2), in which I have shown P_0 slightly displaced from the plane $z=z_m$; this is to emphasise that the z -dependence of G comes into the integrand in (2.4).

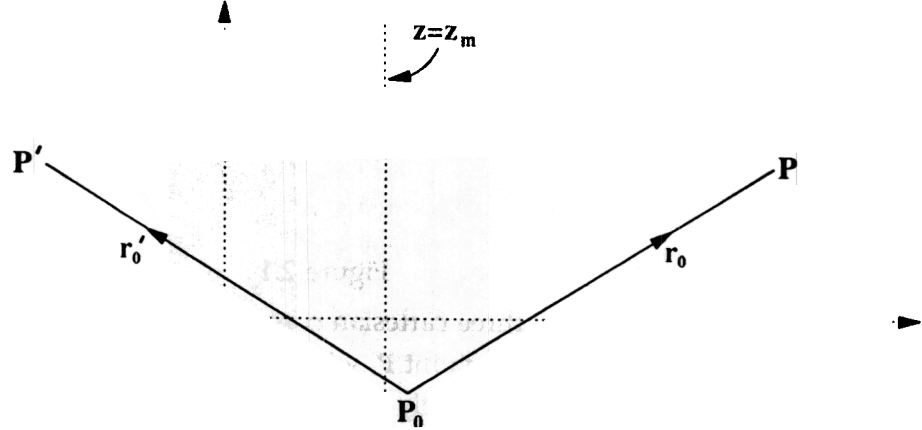


Figure 2.2

The Rayleigh-Sommerfeld Green's function is appropriate to the planar problem because it is equal to zero everywhere on the plane $z=z_m$; with this substitution, equation (2.4) becomes

$$e(x, y, z \geq z_m) = -\frac{1}{4\pi} \iint_{z_0=z_m} e(x_0, y_0, z_0) \frac{\partial G}{\partial z_0} dx_0 dy_0 \quad (2.5)$$

which is what we want: namely, an integral transform from e on the plane $z=z_0$ to e in the half-space $z \geq z_0$. We can now see the advantage of choosing a Green's function which is appropriate to our measurement surface. Equation (2.5) allows us to compute the far field from a knowledge of the near field alone. If we used instead the Kirchoff Green's function e^{ikr}/r , we would be required to provide as data both the field and its normal derivative over the measurement plane. In addition to the practical inconvenience, this would also be theoretically unsatisfactory because it may be shown (see e.g. Born and Wolf 1980, p.377 and references therein) that knowledge of either e or its normal derivative over the measurement plane is sufficient to determine the behaviour of e throughout the half-space $z \geq z_0$; the problem would be overdetermined.

Carrying-out the differentiation with respect to z_0 :

$$e(r, \theta, \phi) = \frac{k_0}{2\pi i} \iint_{z_0=z_m} e(x_0, y_0, z_0) \cos(\theta_0) \frac{e^{ik_0 r_0}}{r_0} \left(1 - \frac{1}{ik_0 r_0}\right) dx_0 dy_0 \quad (2.6)$$

where r, θ, ϕ are spherical-polar far-field coordinates, based on the z -axis, and where θ_0 is the angle between the vector r_0 and the z -axis (figure 2.3). (Note that r_0, θ_0 vary with x_0, y_0 , the coordinates in the measurement plane $z_0=z_m$.) The factor $\cos(\theta_0)$ arises in taking the derivative with respect to z_0 of a function of r_0 .

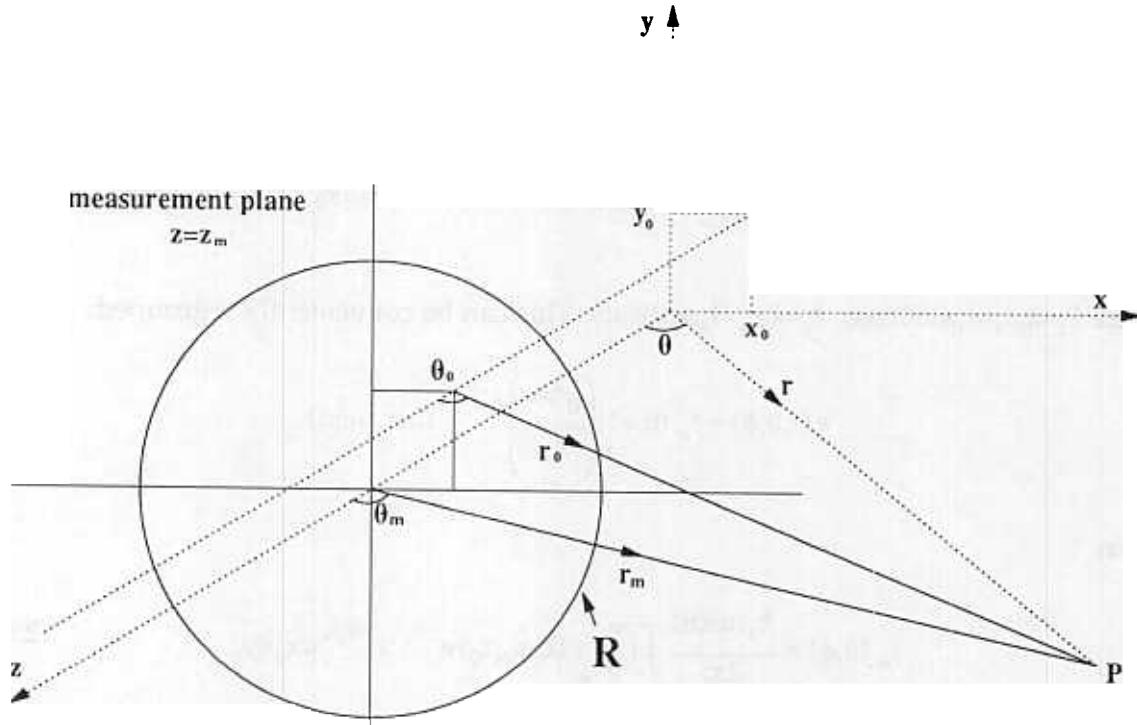


Figure 2.3

Equation (2.6) is exact. For r_0 much greater than $1/k_0$, it reduces to

$$e(r, \theta, \phi) = \frac{k_0}{2\pi i} \iint_{z_0=z_m} e(x_0, y_0, z_0) \cos(\theta_0) \frac{e^{ik_0 r_0}}{r_0} dx_0 dy_0 \quad (r_0 \gg \frac{1}{k_0}) \quad (2.7)$$

Equation (2.7) may in principle be used for calculating the far field from a measured near-field scan. However, the integrand may first be considerably simplified by taking the term $(\cos(\theta_0)/r_0)$ outside the integral. We can do this if r is large enough to ensure that the radiated field is paraxial, *i.e.* that $\cos(\theta_0)/r_0$ varies only negligibly over the region of integration (the far-field condition).

For the far-field condition to be satisfied, we must assume that the field $e(x_0, y_0, z_0)$ has an appreciable amplitude only in some region R with finite area in the centre of the plane $z_0=z_m$ (figure 2.3). The far-field condition will then be satisfied if r_0 is much greater than the greatest length encompassed by R ; equation (2.7) becomes

$$e(r, \theta, \phi) = \frac{k_0}{2\pi i} \frac{\cos(\theta_m)}{r_m} \iint_{z_0=z_m}^R e(x_0, y_0, z_0) e^{ik_0 r_0} dx_0 dy_0 \quad (\text{far field})$$

where r_m, θ_m are the values of r_0, θ_0 respectively in the centre of the $z=z_m$ plane.

As r_0 increases, $r_0/r_m \rightarrow 1$. In the far field, we can expand r_0/r_m to first order in $x_0/r_m, y_0/r_m$:

$$\frac{r_0}{r_m} = 1 - \left(c_x \frac{x_0}{r_m} + c_y \frac{y_0}{r_m} \right) + O \left(\frac{x_0}{r_m} \right)^2, O \left(\frac{y_0}{r_m} \right)^2$$

where c_x, c_y are direction cosines of r_m . Equation (2.8) can now be rewritten:

$$e(r, \theta, \phi) = \frac{k_0}{2\pi i} \frac{\cos(\theta_m)}{r_m} \iint_{z_0=z_m}^R e(x_0, y_0, z_0) e^{-ik_x x_0} e^{-ik_y y_0} dx_0 dy_0 \quad (\text{far field})$$

where $k_x = k_0 c_x = k_0 \sin \theta \cos \phi$, $k_y = k_0 c_y = k_0 \sin \theta \sin \phi$. This can be conveniently regrouped:

$$e(r, \theta, \phi) = f_m(\theta, \phi) \left(\frac{e^{ik_0 r_m}}{r_m} \right) \quad (\text{far field})$$

where

$$f_m(\theta, \phi) = \frac{k_0 \cos(\theta)}{2\pi i} \iint_{z_0=z_m}^R e(x_0, y_0, z_0) e^{-ik_x x_0} e^{-ik_y y_0} dx_0 dy_0 \quad (2.9b)$$

Here, $\cos(\theta_m)$ has been replaced with $\cos(\theta)$; they become equal in the far-field limit.

The function $f_m(\theta, \phi)$ is the far field of the radiating aperture; equation (2.9) shows how the far field is related to the fourier transform of the near-field measured on a plane.

No real antenna exactly satisfies the far-field condition used in deriving (2.9); all antennas which radiate forwards also radiate sideways, albeit at very low power levels. We can only expect the formula to work well if the far-field approximation is good and we must expect errors in the calculated far field due to truncation of $e(x, y, z_m)$ at the edge of R , especially as θ approaches 90° .

2.2 Relation of far field to plane-wave spectrum

The plane waves $e^{i(k,r)}$, with $k=k_0$, form an orthogonal set of functions which completely spans the space of solutions to the Helmholtz equation (2.3). We can therefore expand $e(r)$ in $z \geq z_m$ as a sum of plane waves:

$$e(r) = \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y s(k_x, k_y) e^{i(k,r)} \quad (2.10)$$

where $k=(k_x, k_y, k_z)$. The plane-wave spectrum s is written as a function of k_x, k_y only because k_z is a dependent variable, given by

$$k_z = \sqrt{k_0^2 - (k_x^2 + k_y^2)} \quad (k_x^2 + k_y^2 \leq k_0^2)$$

The interpretation of $s(k_x, k_y)$ when $k_x^2 + k_y^2 > k_0^2$ is discussed in section 2.4 below.

Set $z=z_m$ in (2.10):

$$e(x, y, z_m) = \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y s(k_x, k_y) e^{ik_z(k_x, k_y) z_m} e^{ik_x x} e^{ik_y y}$$

This tells us that $e(x, y, z_m)$ is the fourier transform of $s(k_x, k_y) \exp(ik_z(k_x, k_y) z_m)$. The inverse relation is

$$s(k_x, k_y) e^{ik_z(k_x, k_y) z_m} = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dx_0 \int_{-\infty}^{\infty} dy_0 e(x_0, y_0, z_m) e^{-ik_x x_0} e^{-ik_y y_0}$$

Comparison with equation (2.9b) shows that

$$f_m(\theta, \phi) = -2\pi i k_0 \cos(\theta) s(k_x, k_y) e^{ik_z(k_x, k_y) z_m} \quad (2.11)$$

which tells us that the far field $f_m(\theta, \phi)$ is proportional to the product of the plane-wave spectrum $s(k_x, k_y)$, an inclination factor $\cos(\theta)$ and a non-uniform phase factor $\exp(ik_z(k_x, k_y) z_m)$.

To understand the inclination factor, set z_m equal to zero in order to remove the complicating phase factor, and then write equation (2.10) in the form

$$e(r) = \frac{e^{ik.r}}{r} \left\{ r \int_{-\infty}^{\infty} dk'_x \int_{-\infty}^{\infty} dk'_y s(k'_x, k'_y) e^{i(k'-k).r} \right\}$$

from which, by comparison with equation (2.9a), we can extract for f_m the expression

$$f_m(\theta, \phi) = \lim_{r \rightarrow \infty} r \int_{-\infty}^{\infty} dk'_x \int_{-\infty}^{\infty} dk'_y s(k'_x, k'_y) e^{i(k'-k).r} \quad (2.12)$$

We now observe that, in the limit $r \rightarrow \infty$, the phase of the kernel $\exp(i(k'-k).r)$ of equation

(2.12) starts to vary rapidly with phase, except when $k'=k$, in which case it always has the value 1, however big r becomes. Therefore, in this limit, contributions to the integral cancel, except when $k'=k$, and we can take s outside the integral:

$$f_m(\theta, \phi) = s(k_x, k_y) \lim(r \rightarrow \infty) \left\{ r \int_{-\infty}^{\infty} dk'_x \int_{-\infty}^{\infty} dk'_y e^{i(k'-k) \cdot r} \right\} \quad (2.13)$$

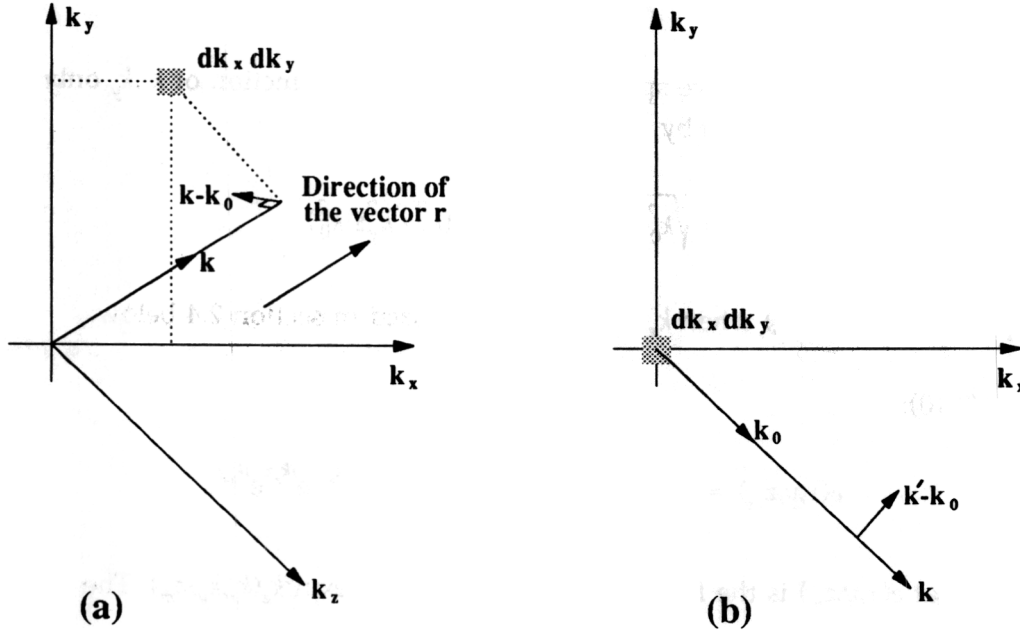


Figure 2.4

Figure 2.4a illustrates the integration of equation (2.13) and figure 2.4b shows that, by rotating the vector k through an angle θ so that it lies along the k_z -axis, we merely introduce a factor $\cos(\theta)$ (the inclination factor) into the equation:

$$\lim(r \rightarrow \infty) \int_{-\infty}^{\infty} dk'_x \int_{-\infty}^{\infty} dk'_y e^{i(k'-k) \cdot r} = \cos(\theta) \cdot \lim(r \rightarrow \infty) \int_{-\infty}^{\infty} dk'_x \int_{-\infty}^{\infty} dk'_y e^{i(k'-k_0) \cdot r} \quad (2.14)$$

where $k_0 = (0, 0, k_0)$. The inclination factor $\cos(\theta)$ is the relative magnitude of the projection $dk_x dk_y$ onto the $k_x k_y$ -plane of a fixed differential area of the surface of the sphere $k = k_0$. It arises because the far-field $f_m(\theta, \phi)$ is defined in terms of a differential $d\theta d\phi$ which remains constant as k moves over the far-field sphere, whereas the plane-wave spectrum $s(k_x, k_y)$ is defined in terms of the differential $dk_x dk_y$, which varies as $\cos(\theta)$.

The integral on the right-hand side of equation (2.14) is a universal function of k_0 and r , and can be evaluated by setting

$$|k' - k_0| = \theta' k_0$$

and changing the variables of integration to θ' (figure 2.5):

$$\int_{-\infty}^{\infty} dk'_x \int_{-\infty}^{\infty} dk'_y e^{i(k'-k_0) \cdot r} = 2\pi k_0^2 \int_0^{\Delta\theta} \theta' e^{i\left(k_0 r \frac{\theta'^2}{2}\right)} d\theta'$$

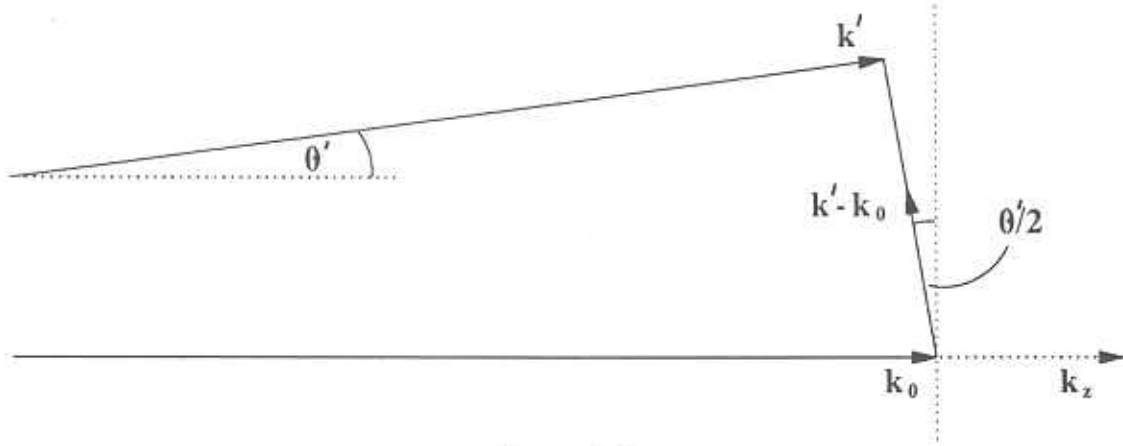


Figure 2.5

where the top limit $\Delta\theta'$ can be made arbitrarily small as $r \rightarrow \infty$. Integrating, and noting that it is only at the lower limit, $\theta' = 0$, that the integrand contributes as $r \rightarrow \infty$:

$$\begin{aligned} \int_{-\infty}^{\infty} dk'_x \int_{-\infty}^{\infty} dk'_y e^{i(k' - k_0) \cdot r} &= \frac{2\pi k_0}{ir} \left[e^{i(k_0 r \theta'^2/2)} \right]_0^{\Delta\theta'} \\ &= -\frac{2\pi i k_0}{r} \end{aligned}$$

Substitution of this expression into equation (2.13) gives equation (2.11), as required for consistency.

2.3 Dependence of the calculated far field on z_m

The far field $f_m(\theta, \phi)$ is defined (equation 2.9a) such that its phase will be constant in the case of an emitter of spherical waves, located at the point $(0, 0, z_m)$ in the centre of the measurement plane. In general, the phase centre of our source will not lie in the measurement plane. Consider an emitter of spherical waves located at the origin: the phase of f_m will vary by $+k_0 z_m$ as θ is reduced from 90° to 0° . This can be seen explicitly in equation (2.11).

With this z_m -variation in mind, we define a generalised far field $f(\theta, \phi)$ as follows

$$f(\theta, \phi) = f_m(\theta, \phi) e^{-ik_z(k_x, k_y) z_m} \quad \text{where} \quad \begin{cases} k_z = \sqrt{k_0^2 - (k_x^2 + k_y^2)}, \\ k_x = k_0 \sin \theta \cos \phi, \\ k_y = k_0 \sin \theta \sin \phi \end{cases} \quad (2.15)$$

The relation between f and s is:

$$f(\theta, \phi) = -2\pi i k_0 \cos \theta s(k_x, k_y) \quad (2.16)$$

2.4 Evanescent waves

Equation (2.16) relates the far-field pattern f to the plane-wave spectrum s within the circular region of k_x, k_y -space $|(k_x, k_y)| < k_0$. Points on the circle correspond to points on the edge ($\theta=90^\circ$) of the far-field pattern; the origin of (k_x, k_y) -space corresponds to radiation in the boresight direction.

Outside this circle, we have

$$k_x^2 + k_y^2 > k_0^2$$

from which it follows that k_z is imaginary. The plane-wave solutions to equation (2.3) now take the form:

$$e \sim e^{ik_x x} e^{ik_y y} e^{-\alpha z} \quad (\alpha = \sqrt{k_x^2 + k_y^2 - k_0^2})$$

We discard the set of solutions with $\alpha < 0$ because these correspond to the physically impossible phenomenon of waves which get stronger as we get further from the source antenna.

Plane waves of the form (2.17) are called evanescent waves because they decay away as z is increased. In a transverse near-field scan, the presence of evanescent waves will be indicated by oscillations with wavelength less than the free-space wavelength $2\pi/k_0$. The heights of the peaks in $s(k_x, k_y)$ corresponding to evanescent modes will die away with z as $e^{-\alpha z}$. The decay rate α is zero on the circle $|(k_x, k_y)| = k_0$ and it increases with increasing $|(k_x, k_y)|$. The rapidity of transverse oscillation is therefore linked, through α , to the rate of decay with z : the more rapid the oscillations, the more rapid the decay.

2.5 Scalar transformation: summary of formulas

Let

$$\tilde{e}_m(k_x, k_y) = \frac{1}{2\pi} \iint_{z_0=z_m}^R e(x_0, y_0, z_0) e^{-ik_x x_0} e^{-ik_y y_0} dx_0 dy_0 \quad (2.18)$$

Then the far field $f(\theta, \phi)$ is given by

$$f(\theta, \phi) = -ik_0 \cos \theta e^{-ik_x(k_x, k_y)z_m} \tilde{e}_m(k_x, k_y) \quad (2.19)$$

where

- z_m is the z -coordinate of the measurement plane,
- R is the region of the measurement plane over which the field has appreciable amplitude,
- θ, ϕ are spherical polar coordinates based on the z -axis,

$$\begin{aligned}
k_x &= k_0 \sin \theta \cos \phi, \\
k_y &= k_0 \sin \theta \sin \phi, \\
k_z &= \sqrt{k_0^2 - (k_x^2 + k_y^2)}.
\end{aligned}$$

The plane wave spectrum is defined, for all values of k_x, k_y by

$$s(k_x, k_y) = \frac{1}{2\pi} e^{ik_x(k_x, k_y)z_m} \tilde{e}_m(k_x, k_y) \quad (2.20)$$

and s , in the region $|(k_x, k_y)| < k_0$, is related to f by:

$$f(\theta, \phi) = -2\pi i k_0 \cos \theta s(k_x, k_y).$$

In the region $|(k_x, k_y)| \geq k_0$, s represents evanescent modes of the form

$$e \sim e^{ik_x x} e^{ik_y y} e^{-\alpha z} \quad (\alpha = \sqrt{k_x^2 + k_y^2 - k_0^2})$$

Some care must be taken in scaling the output of numerical fourier transform routines. The NAG routine C06FUF, for example, returns the value of the expression:

$$\frac{1}{\Delta x_0 \Delta y_0 \sqrt{MN}} \iint_{z_0=z_m} e(x_0, y_0, z_0) e^{-ik_x x_0} e^{-ik_y y_0} dx_0 dy_0$$

where $\Delta x, \Delta y$ are the distances between successive data points, and where M and N are the number of grid points parallel to the x and y axes, respectively. Output from this routine therefore needs to be scaled by multiplying by the factor

$$\frac{\Delta x \Delta y \sqrt{MN}}{2\pi}.$$

Another possible source of confusion in implementing (2.18) numerically is the position of the origin of the data set. The NAG routine already mentioned takes the the origin to be one corner of the grid, whereas the natural origin for an experimental data set is the centre. The following general property of fourier transforms:

$$F_k \{f(x-x_s)\} = e^{-ikx_s} F_k \{f(x)\}$$

where $F_k\{\}$ represents taking the fourier transform from x -space into k -space, allows us to correct for this discrepancy. We simply increase, k -point by k -point, the phase of the output by

$$(k_x x_s + k_y y_s)$$

where (x_s, y_s) is the position of the centre of the data set, relative to the origin assumed by the fourier transform (x_s and y_s are positive in the case of the routine C06FUF, which takes the bottom left corner of the measurement grid as its origin). This correction is formally identical to the phase correction introduced in section 2.3 above to take account of the z -coordinate of the measurement plane.

2.6 Scalar transformation: how to deduce antenna gain

The near field e gives rise to a travelling wave, total power P_m , in the probe feed (figure 2.6). We measure a voltage V_m , with modulus proportional to the square root of P_m , and a voltage V_0 , with modulus proportional to the square root of P_0 , the power entering antenna A.

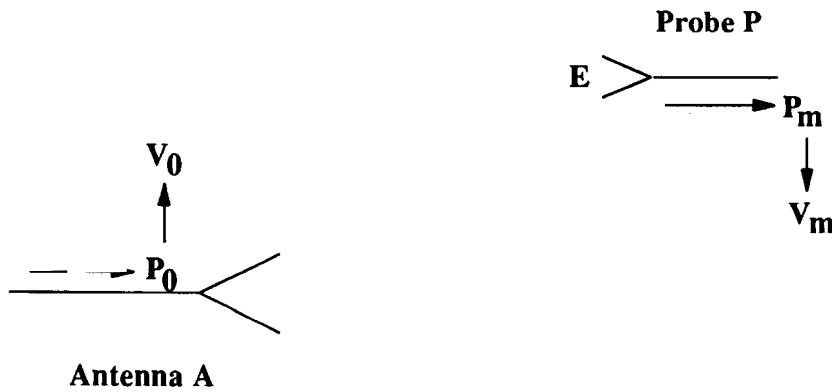


Figure 2.6: the near-field measurement

Define α and β by

$$\begin{aligned} V_0^2 &= \alpha^2 P_0 \\ V_m^2 &= \beta^2 P_m \end{aligned}$$

In a plane wave, P_m is directly proportional to e^2 and we use the effective area formula (Silver 1984, section 2.14) to derive an expression for P_m in terms of the plane-wave electric field e_{PW} :

$$P_m = \gamma^2 e_{PW}^2 \quad \left(\gamma^2 = \frac{\lambda^2 \epsilon_0 c G_P}{4\pi} \right)$$

The k -space quantities $\hat{e}$ and V_m are therefore related by

$$\hat{V}_m = \beta \gamma \hat{e}$$

(Symbols refer to the modulus of the RMS value throughout.)

A separate normalisation (straight-through) measurement is performed (figure 2.7), with antenna and probe feeds joined together directly. From the above relations, we have

$$\begin{aligned} V_0'^2 &= \alpha^2 P_0' \\ V_m'^2 &= \beta^2 P_0' \end{aligned}$$

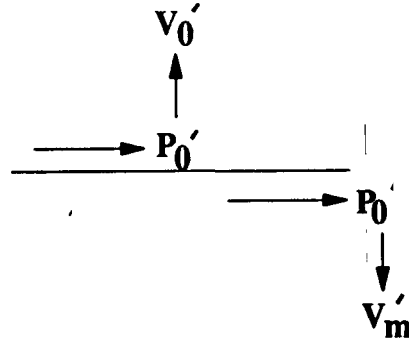


Figure 2.7: The normalisation measurement

In each case, we divide V_m by V_0 :

$$\frac{V_m}{V_0} = \frac{\beta}{\alpha} \left(\frac{P_m}{P_0} \right)^{\frac{1}{2}}$$

$$\frac{V_m'}{V_0'} = \frac{\beta}{\alpha}$$

The gain G of the antenna under test is given by

$$\frac{P_0 G_A}{4\pi r^2} = \epsilon_0 c \left(\frac{f}{r} \right)^2 = \epsilon_0 c \left(\frac{k_0 \cos\theta \hat{e}}{r} \right)^2$$

(using 2.19), with f and $\hat{e}$ referring to the radiation direction of interest. On boresight,

$$G_A = \delta \frac{\hat{e}^2}{P_0} \quad (\delta = 4\pi\epsilon_0 c k_0^2)$$

$$= \delta \frac{\hat{e}^2}{V_m^2} \frac{V_m^2}{V_0^2} \frac{V_0^2}{P_0}$$

$$\frac{\alpha^2 \delta}{\beta^2 \gamma^2} \left(\frac{V_m}{V_0} \right)^2$$

$$\frac{\delta}{\gamma^2} \left(\frac{\hat{V}_m}{V_0} \frac{V'_0}{V'_m} \right)^2$$

$$= \frac{64\pi^4}{\lambda^4 G_p} \left(\frac{\hat{V}_m}{V_0} \frac{V'_0}{V'_m} \right)^2$$

(inserting the expression derived above for γ^2).

2.7 The vector far field

By applying the scalar transformation formula to x - and y -components of the measured near field, we can obtain the x - and y -components of the far field. By definition, waves in the far field are spherical, centred on the origin. The propagation direction unit vector u_k is normal to the surface of the far-field sphere and the resultant electric field vector E lies in this surface, perpendicular to u_k . We can therefore write $E \cdot u_k = 0$ or

$$E_x u_x + E_y u_y + E_z u_z = 0 .$$

Given a direction vector u and the corresponding scalar fields E_x, E_y , the above equation determines E_z .

To determine the vector far field, it is therefore necessary to measure and transform only the transverse components E_x and E_y of the near field.

As a basis for expressing the polarization of the vector far field, it is natural to choose a spherical coordinate system, because the radial component of the field is always zero and the polarization vector can therefore be expressed using only two numbers, instead of the three necessary in general. The two transverse polarization directions are usually known as co-polar and cross-polar. For the purposes of discussing various alternative coordinate systems, these labels will be used in such a way that, for a wave propagating along the boresight line $0z$, a purely co-polar field points in the $0y$ direction, and a purely cross-polar field points in the $0x$ -direction. This will be true for all coordinate systems; the differences emerge as we leave the boresight line and look further out in the angular far-field pattern.

Consider first the spherical polar system based on the z -axis. The angular coordinates (u_θ, u_ϕ) of this system are good for specifying the propagation direction of waves in the far field (figure 2.8). However, they are not good for specifying the polarization of the far field because the definitions of the polarization unit vectors u_θ, u_ϕ break down as the propagation direction approaches the boresight line $0z$, usually the most important region of the far-field pattern, and on the boresight it is impossible to define co- and cross-polar

at all in terms of u_θ and u_ϕ . It is this definition of co- and cross-polar which corresponds exactly to the fields measured in the pattern measurement method used here at NPL, and we can use this fact to illustrate the nature of the problem on boresight. What are we measuring as we roll the antenna under test around its boresight axis when it is pointing directly at the probe - E_θ or E_ϕ ? The answer is: neither; we need to supplement our description of polarization at this point by making use of the vectors u_x and u_y . This makes spherical polars based on the z-axis a bad basis for specifying polarization.

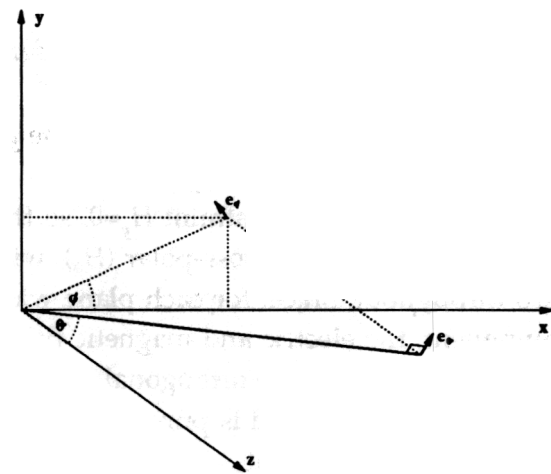


Figure 2.8: Spherical polars based on the z-axis

Spherical polars work much better if they are based on the x-axis (figure 2.9), with u_ϕ as the co-polar direction and u_θ as cross-polar. If our source antenna is a horn fed by rectangular waveguide, then this basis is especially appropriate for the following reason. Suppose the horn was ideal in the sense that it produced a near field in which $E_x=0$ all over the measurement plane; then E_x will be zero everywhere (equation 2.3). Now u_ϕ is everywhere orthogonal to u_x and u_θ is linearly dependent on u_x except in the unimportant cases $\theta=0,\pi$ (radiation along the $\pm x$ axes). The far field is therefore purely co-polar (E_ϕ) in this case, with cross-polar component E_θ equal to zero.

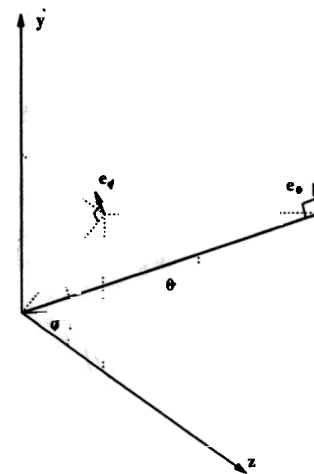


Figure 2.9: spherical polars based on the x-axis

The unit polarization vectors u_θ, u_ϕ are defined in this coordinate system by the equations:

$$\begin{aligned} |u_\theta| &= |u_\phi| = 1 \\ u_\phi \cdot u_x &= u_\phi \cdot u_r = u_\theta \cdot u_r = u_\theta \cdot u_\phi = 0 \end{aligned}$$

where $u_r = (k_x, k_y, k_z)$ ($k_x^2 + k_y^2 + k_z^2 = 1$). It follows that

$$\begin{aligned} u_x &= (1 - k_x^2)^{-\frac{1}{2}} (0, k_z, -k_y) = -u_\phi \\ u_\theta &= (1 - k_x^2)^{-\frac{1}{2}} (-(k_y^2 + k_z^2), k_x k_y, k_z k_x) = -u_\phi \end{aligned}$$

An alternative, which exploits in a similar way the separability of the Helmholtz equation in cartesian coordinates is spherical polars based on the y-axis, with u_θ as the co-polar direction and u_ϕ as cross-polar (figure 2.10). In this system, a near field with zero y-

component gives a cross-polar far field everywhere. This choice is appropriate for describing the far field from an array of Hertzian dipoles, oriented parallel to the y -axis; radiation from such dipoles is characterised by the constraint $H_y=0$, so the far magnetic field is pure cross-polar (H_ϕ), with H_θ zero throughout. Since, for each plane-wave component, the electric and magnetic field polarization vectors are orthogonal, this means that the electric far field is pure co-polar (E_θ), with E_ϕ zero throughout.

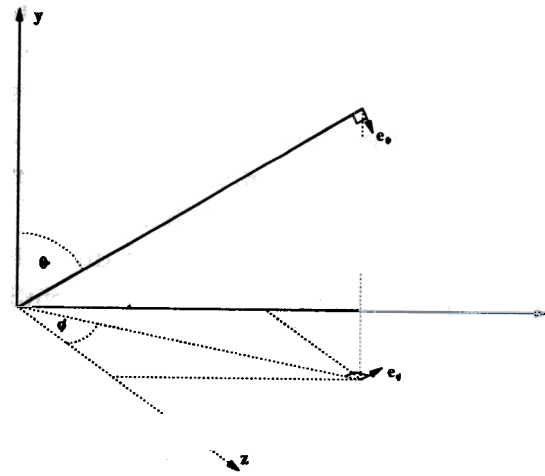


Figure 2.10: spherical polars based on the

y -axis
The unit polarization vectors u_θ, u_ϕ are defined in this coordinate system by the equations:

$$|u_\theta| = |u_\phi| = 1$$

$$u_\phi \cdot u_y = u_\phi \cdot u_r = u_\theta \cdot u_r = u_\theta \cdot u_\phi = 0$$

from which it follows that their relation to the corresponding k -space coordinates (k_x, k_y, k_z) (where $k_x^2 + k_y^2 + k_z^2 = 1$) is

$$u_\phi = (1 - k_y^2)^{-1/2} (k_z, 0, -k_x) = u_{cr}$$

$$u_\theta = (1 - k_y^2)^{-1/2} (k_x k_y, -(k_x^2 + k_z^2), k_z k_y) = -u_{co}$$

Both x - and y -based spherical polars have the advantage over any other spherical coordinate system that, if one cartesian component of the transverse near field is zero, then one transverse component of the far field is zero.

Ludwig (1973) has surveyed several coordinate systems commonly used as bases for writing down far-field polarization. The two spherical polar systems I have already described, based on the x and y axes, are the two possible varieties of Ludwig's second definition (L2 for short). Ludwig recommends a different coordinate system L3. The L3 co- and cross-polar vectors (Ludwig, eq. 8) can be specified implicitly by the following fundamental property: in L3, any purely co-polar far-field pattern becomes purely cross-polar if the source antenna is rotated through 90° about the z -axis, for example by the insertion of a section of twisted waveguide in some part of the feed, lying along Oz . The L3 co- and cross-polar fields correspond exactly to the fields measured in the pattern measurement method described by Silver (1949, pp 561-4), but this is different from the NPL measurement method.

The unit polarization vectors u_{co}, u_{cr} are defined in L3 by the equations:

$$\begin{aligned}
|u_{co}| &= |u_{cr}| = 1 \\
u_{co} \cdot u_r &= u_{cr} \cdot u_r = u_{co} \cdot u_{cr} = 0 \\
u_{co} \cdot u_\phi^{0z} &= \cos(\phi)
\end{aligned}$$

where u_ϕ^{0z} is the unit vector pointing in the direction of increasing azimuth in the spherical polar coordinate system based on the z-axis. It follows that their relation to the corresponding k -space coordinates (k_x, k_y, k_z) (where $k_x^2 + k_y^2 + k_z^2 = 1$) is

$$u_{co} = -\frac{k_x k_y}{1+k_z}, 1 - \frac{k_y^2}{1+k_z}, -k_y$$

$$u_{cr} = \frac{k_x^2}{1+k_z}, \frac{k_x k_y}{1+k_z}, -k_z$$

Once we have chosen an appropriate polarization basis and derived, in terms of k_x, k_y and k_z , expressions for the Cartesian components of the co- and cross-polar unit vectors, it is a straightforward matter to re-express the far field, as calculated in Cartesian components, on the new basis. The co-polar component is given by

$$\begin{aligned}
E_{co} &= (u_{co} \cdot E) \\
&= (u_{co})_x E_x + (u_{co})_y E_y + (u_{co})_z E_z \\
&= (u_{co})_x E_x + (u_{co})_y E_y - \frac{(u_{co})_z}{k_z} (E_x k_x + E_y k_y) \\
&= (u_{co})_x - (u_{co})_z \frac{k_x}{k_z} E_x + (u_{co})_y - (u_{co})_z \frac{k_y}{k_z} E_y
\end{aligned}$$

and the cross-polar component by

$$E_{cr} = (u_{cr})_x - (u_{cr})_z \frac{k_x}{k_z} E_x + (u_{cr})_y - (u_{cr})_z \frac{k_y}{k_z} E_y$$

The NPL near-field-to-far-field transformation software allows the user to choose the polarization basis most appropriate to the problem in hand. Wood (1980, pp 123-4) and

Clarricoats and Over 1984 proposed to recommend L3. In the special case $\sin$ probe characterization using ideal thin lens approximation, L3 proved to be the most convenient

L3 the above expressions simplify to

$$E. \quad \frac{y}{k(1+k_z)} \quad \frac{k_y}{k_z(1+k_z)}$$

and

$$\frac{k_x}{k(1+k_z)} \quad \frac{y}{k(1+k_z)}$$

3 PRACTICE

If we could start from a complete and accurate knowledge of the near field $e(x_0, y_0, z_m)$ over the region R of the plane $z=z_m$ in which e has appreciable amplitude, then we would be able to calculate exactly the far field in any direction (θ, ϕ) and the plane wave spectrum at any point (k_x, k_y) , using the formulas derived so far.

In addition to noise and any unknown systematic errors in our measurement system, there are four ways in which our knowledge of the near field is incomplete or distorted.

Firstly, truncation: e never falls to zero on the measurement plane; we scan over a finite region R , chosen such that $|e|$ has fallen below some stipulated fraction of its maximum value at the edge of R .

Secondly, sampling: we measure the field at a finite set of discrete points in R .

Thirdly, probe characteristic: no probe measures field at a point; even the smallest probes such as the optically modulated scatterer (Hygate and Nye 1990) take a spatial average of the field. Our measured data are the convolution of the true field and a probe function.

Fourthly, interaction: all probes work by disturbing the field in which they have been immersed; in other words, they scatter radiation. When the probe is close to the source, the scattered wave it sends out encounters the source and is partially reflected back towards the probe, where it distorts the measured field.

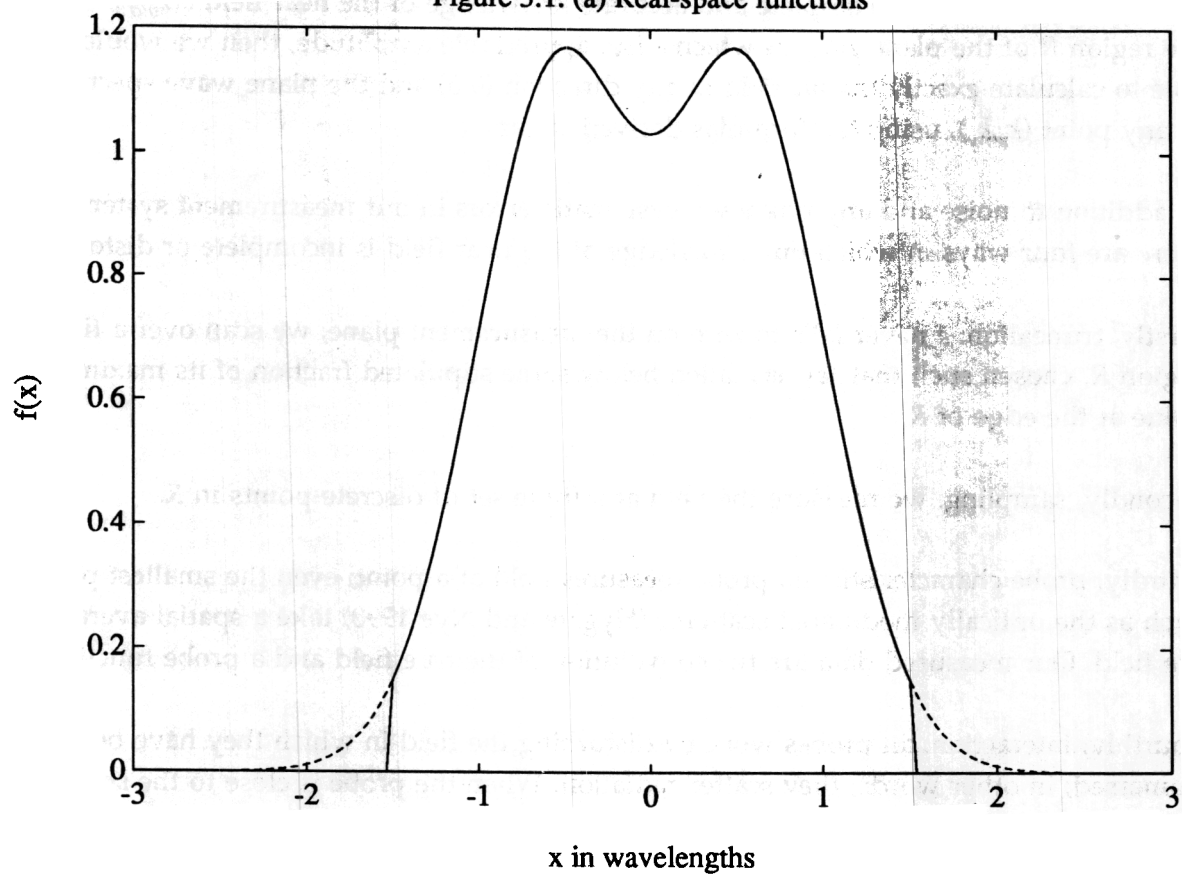
The remainder of this report addresses these four metrological problems. I will confine the treatment to one-dimensional data sets, (functions of x), when the extension to two dimensions is straightforward. I will also make a distinction between the signal (which we are trying to measure) and noise.

3.1 Truncation of the signal

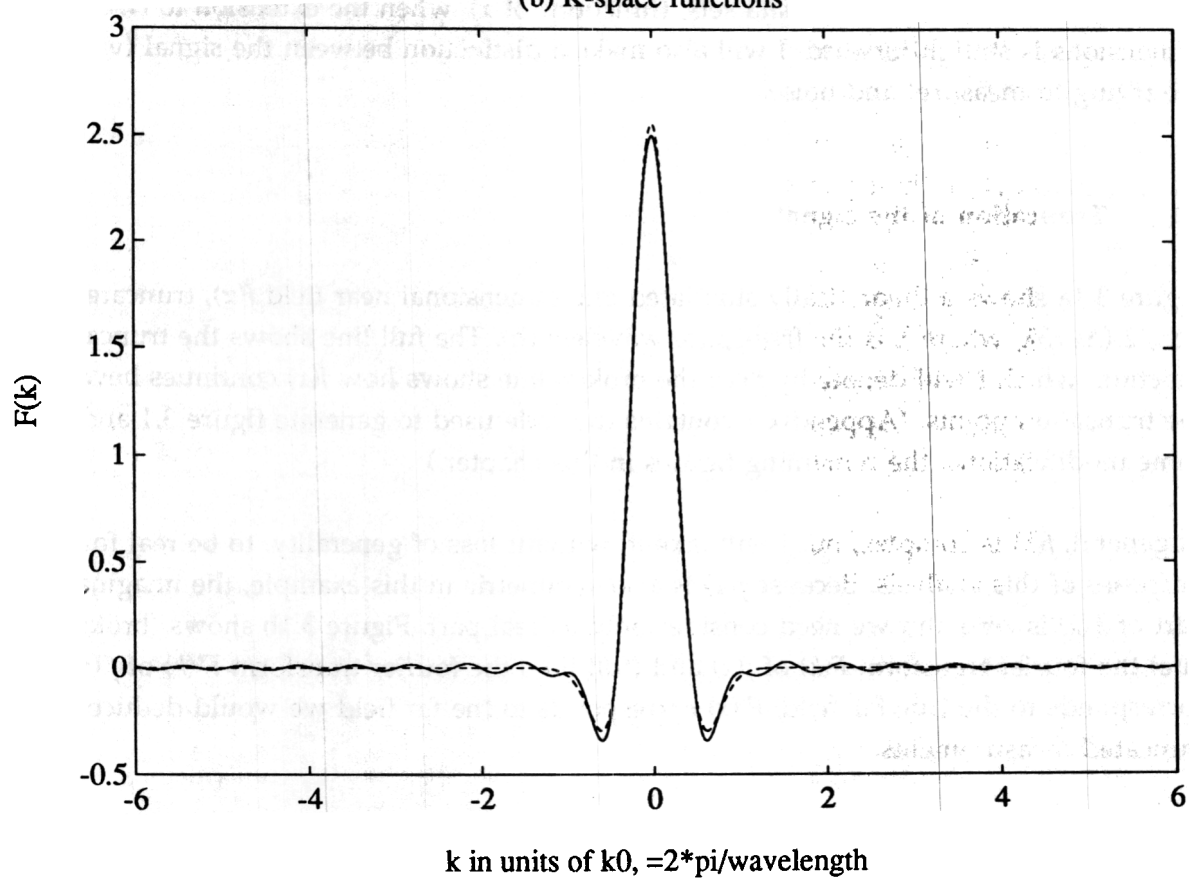
Figure 3.1a shows a theoretically simulated one-dimensional near field $f(x)$, truncated at $\pm\Delta x/2$ ($\Delta x=3\lambda$, where λ is the free-space wavelength). The full line shows the truncated function, which I will denote by $f'(x)$; the broken line shows how $f(x)$ continues beyond the truncation points. (Appendix 1 contains the code used to generate figure 3.1 and, with some modifications, the remaining figures in this chapter.)

In general, $f(x)$ is complex, but I will take it, without loss of generality, to be real for the purposes of this analysis. Because $f(x)$ is also symmetric in this example, the imaginary part of $F(k)$ is zero and we need consider only its real part. Figure 3.1b shows (broken line) the fourier transform $F(k)$ of $f(x)$ and (full line) the fourier transform $F'(k)$ of $f'(x)$. $F(k)$ corresponds to the true far field; $F'(k)$ corresponds to the far field we would deduce, using truncated measurements.

Figure 3.1: (a) Real-space functions



(b) K-space functions



The effect in k -space of real-space truncation is to add to $F(k)$ a ripple, k -wavelength $4\pi/\Delta x$. This depresses the main peak slightly, and increases the height of the two secondary peaks. The ripple dies away very slowly with increasing k , and it is most noticeable in the regions $k > k_0$ ($k_0 = 2\pi/\lambda$), where the true signal is small.

If we truncate our measurement at points where the field still has appreciable magnitude, our far field will be distorted by ripple. (In principle, we could use the magnitude of any such ripple in $F(k)$ as a measure of how badly the real-space signal has been truncated, but in practice, this relation is complicated by noise truncation, which will be dealt with in the next section of this report.)

Suppose we choose our truncation length Δx to be sufficiently large that $f(x)$ is negligible outside $\{-\Delta x/2, \Delta x/2\}$ (e.g. figure 3.2a, in which Δx has been set to 6λ). Then we can write

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\Delta x/2}^{\Delta x/2} f(x) e^{-ikx} dx \quad (3.1)$$

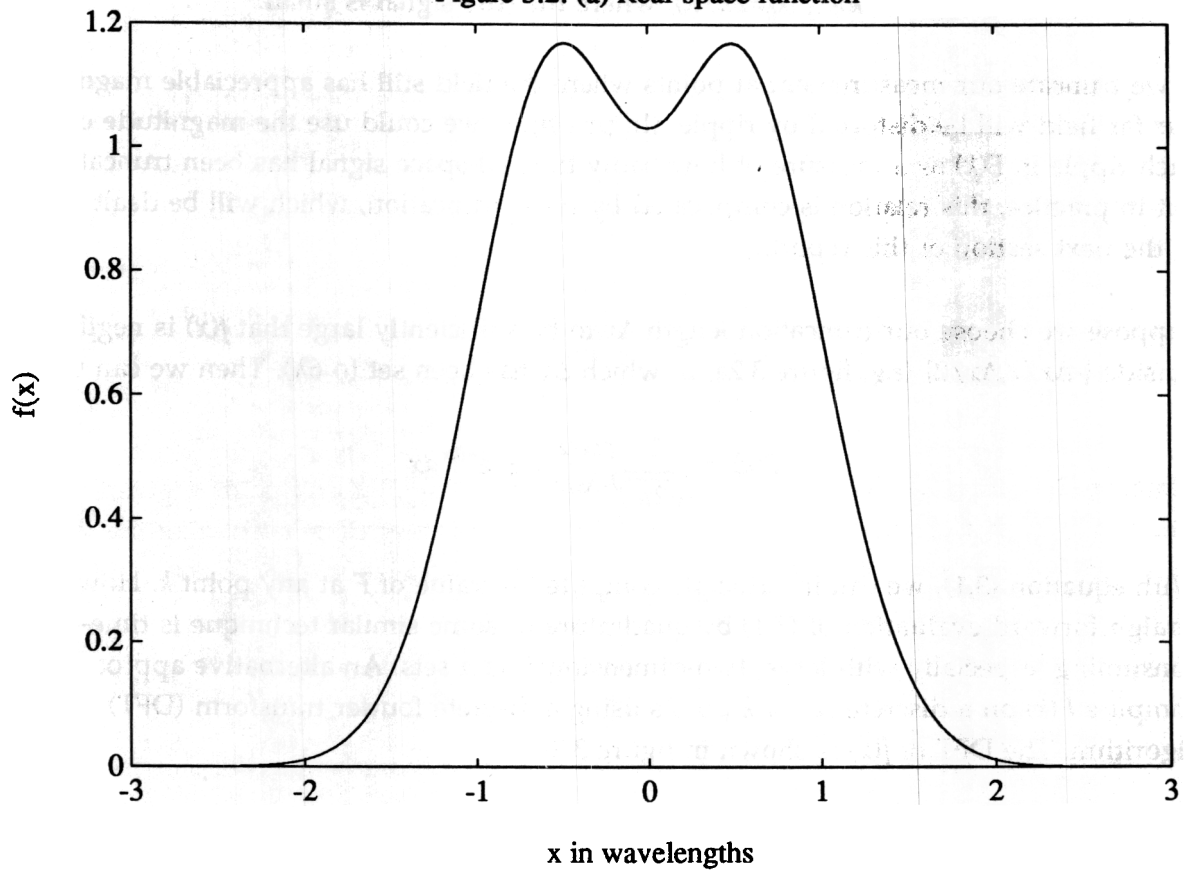
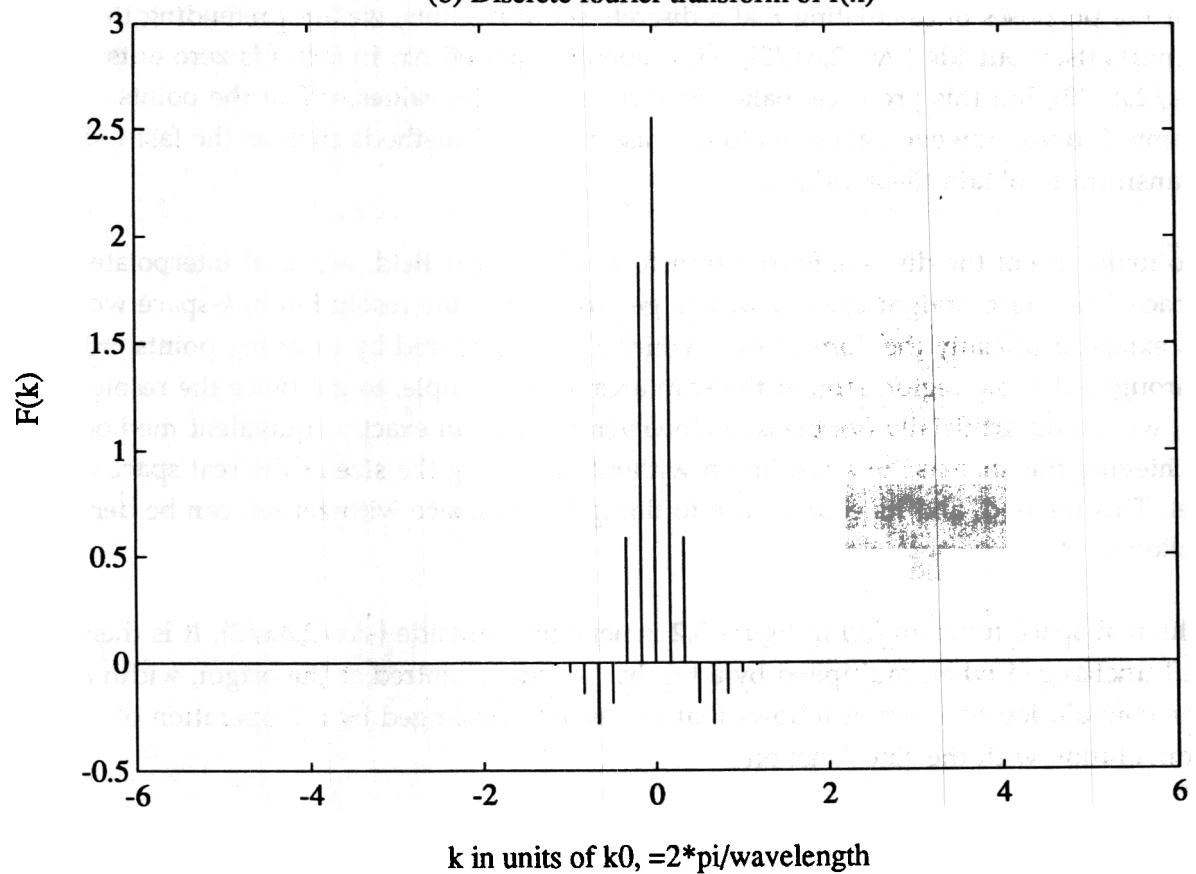
With equation (3.1), we can in principle compute the value of F at any point k . However, straightforward evaluation of (3.1) by quadrature or some similar technique is time-consuming, especially with large, two-dimensional data sets. An alternative approach is to compute $F(k)$ on a discrete set of k -points using a discrete fourier transform (DFT) algorithm. The DFT of $f(x)$ is shown in figure 3.2b.

The DFT is fast, but only because its resolution in k -space is low. The output is the product of $F(k)$ (figure 3.1b, broken line) and an infinite comb of delta functions, spacing $2\pi/\Delta x$. By the convolution theorem, it follows that we can regard the input as the convolution of $f(x)$ and an infinite comb of delta functions, spacing Δx . This means that, for the purposes of calculating F at a discrete set of k -points, we are pretending that $f(x)$ repeats itself outside $\{-\Delta x/2, \Delta x/2\}$, with repetition period Δx . In fact, f is zero outside $\{-\Delta x/2, \Delta x/2\}$, but this pretence makes no difference to the values of F at the points on the comb. It does, however, allow us to use fast numerical methods such as the fast fourier transform to obtain these values.

To make use of the discrete fourier transform of the near field, we must interpolate in k -space. The most straightforward way to get arbitrarily fine resolution in k -space would be to extend artificially the domain over which $f(x)$ is measured by inserting points $f=0$ throughout some region around the scan area. (For example, to get twice the resolution in k_x , we would artificially double Δx .) However, there is an exactly equivalent method of achieving this increase in k -resolution without increasing the size of the real-space data set. This method, and its equivalence to filling-in real space with zeroes, can be derived as follows.

The real-space function $f(x)$ in figure 3.2 is negligible outside $\{-\Delta x/2, \Delta x/2\}$. It is therefore left unchanged when multiplied by a top-hat function, centred at the origin, width Δx . By the convolution theorem, it follows that $F(k)$ is left unchanged by the operation of convolution with the sinc function

Figure 3.2: (a) Real-space function

(b) Discrete fourier transform of $f(x)$ 

$$\frac{\Delta x}{i\sqrt{2\pi}} \frac{\sin\{(\Delta x/2)k\}}{(\Delta x/2)k}$$

which is itself the fourier transform of the top-hat function already described. In algebraic language, we can write that F satisfies the equation

$$F(k) = \hat{s}_{\Delta x}(k, k')F(k')$$

where

$$\hat{s}_{\Delta x} = \frac{\Delta x}{i\sqrt{2\pi}} \int_{-\infty}^{\infty} dk' \frac{\sin[(\Delta x/2)(k-k')]}{(\Delta x/2)(k-k')}$$

Equation (3.1) provides us with an algorithm for computing F at any point k , given the values of $F(k')$ at a discrete set of points k' . There will be an error in estimating the value of the convolution integral because the k' -points are not infinitesimally close together.

To summarize: in the absence of noise, interpolation is straightforward. We choose a scan area big enough to ensure that the signal has fallen to an acceptably low level at the edge; we use a DFT routine to obtain estimates of $F(k)$ at set of discrete k -points; we then interpolate using equation (3.1). The amplitude of any ripple with k -wavelength $4\pi/\Delta x$ in the interpolated curve will indicate the magnitude of any error due to truncation at the edge of the scan area.

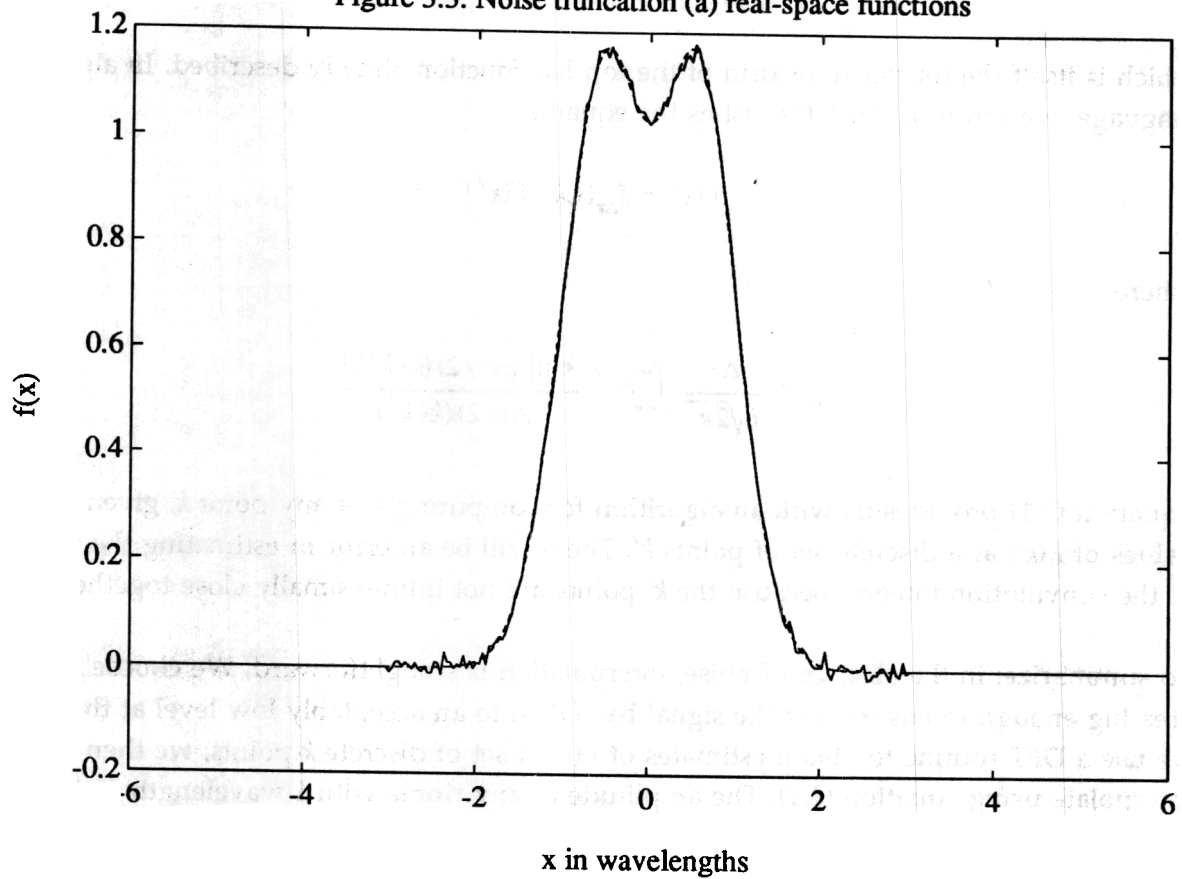
3.2 Truncation of noise

The presence of noise complicates the picture, but the same principles apply. Suppose that we measure the signal throughout the region over which it can be distinguished from the noise. Then, however large we make our truncation area, there will always be noise at the truncation point, and we expect to see the effects of noise truncation as a ripple in k -space.

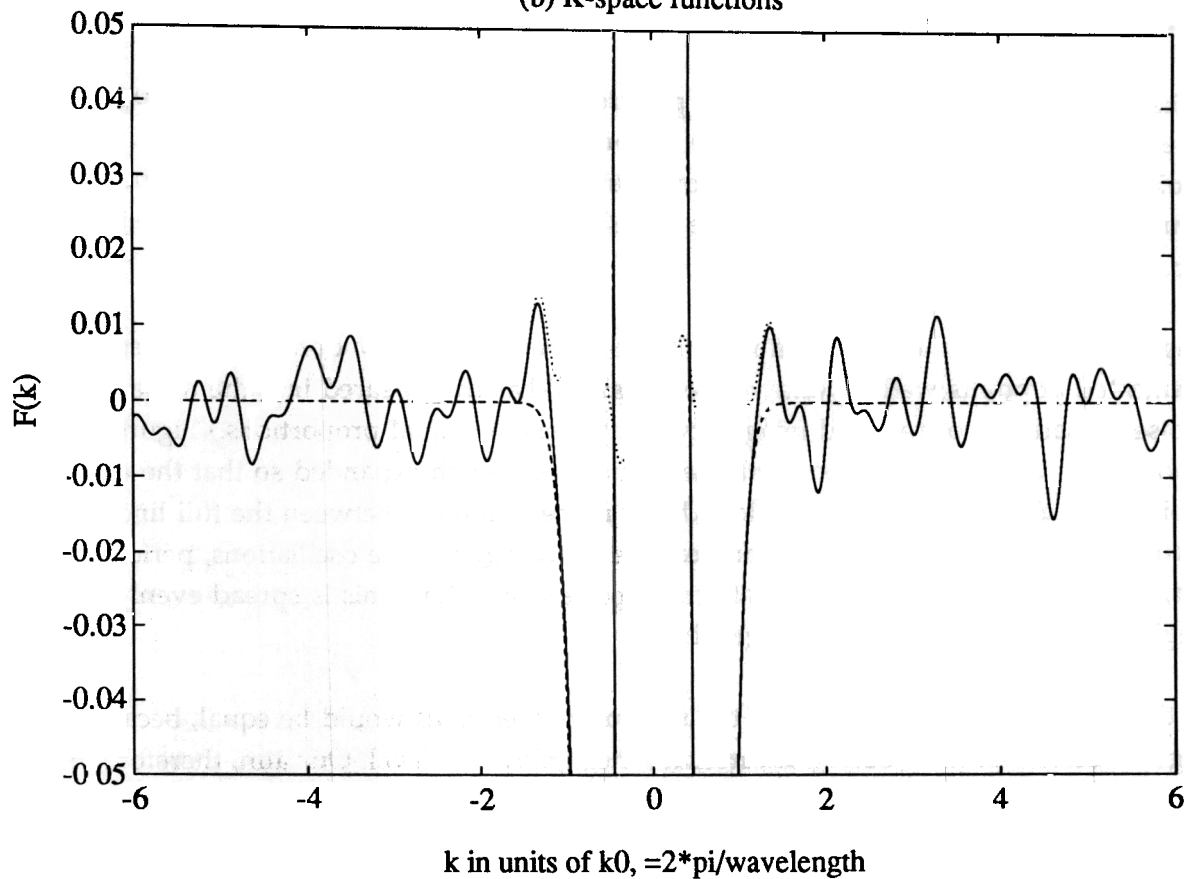
Figure 3.3a shows the simulated near field of figure 3.1 with random noise added, and truncated at points well beyond where the signal has disappeared into the noise. (The noise is made up of real and imaginary components in equal proportions.) Figure 3.3b shows the corresponding $F(k)$, with the vertical axis much expanded so that the effects of noise truncation are visible. The dotted line is the difference between the full line (truncated, noisy signal) and the broken line (pure signal). The oscillations, period about $4\pi/\Delta x$, in the dotted line, are the effects of noise truncation. This is spread evenly over the region of k -space covered by the graph.

If there were no noise in the data, the full and broken lines would be equal, because the signal itself must have been truncated at a negligibly low level. Our aim, therefore, in deriving $F(k)$ from the output of the DFT routine, is to get the best possible estimate of

Figure 3.3: Noise truncation (a) real-space functions



(b) K-space functions



the broken curve. We are starting from a discrete set of evenly-spaced points on the full curve, and, since these points are themselves affected by the noise-truncation ripple, the problem has become one of fitting, rather than simply interpolation. We are trying to satisfy simultaneously three incompatible constraints:

(1) $F(k)$ must pass through all the points generated by the DFT routine,

$$(2) \quad F(k) = \hat{s}_{\Delta x}(k, k') F(k')$$

(3) there must be no ripples of k -wavelength $4\pi/\Delta x$ in $F(k)$

The interpolation method outlined in section (3.1) wholly satisfies criteria (1) and (2) and consequently gives rise to ripples in $F(k)$ due to noise truncation. In order to reduce uncertainty due to these ripples, we need to relax one or both of the first two criteria.

In k -space, the most straightforward way to proceed is to define an error function:

$$\Delta = \delta^2 + \gamma \int_{-\infty}^{\infty} |\varepsilon(k)|^2 dk$$

where

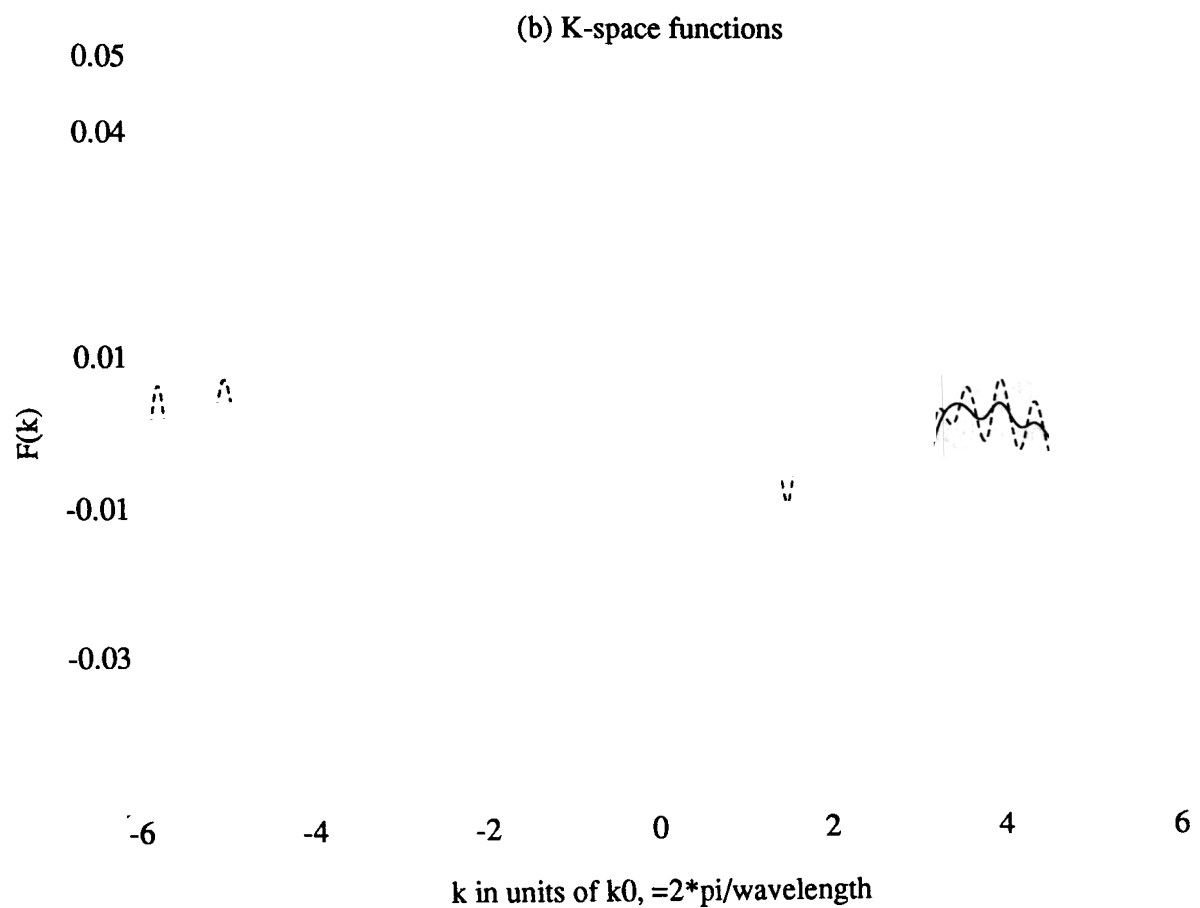
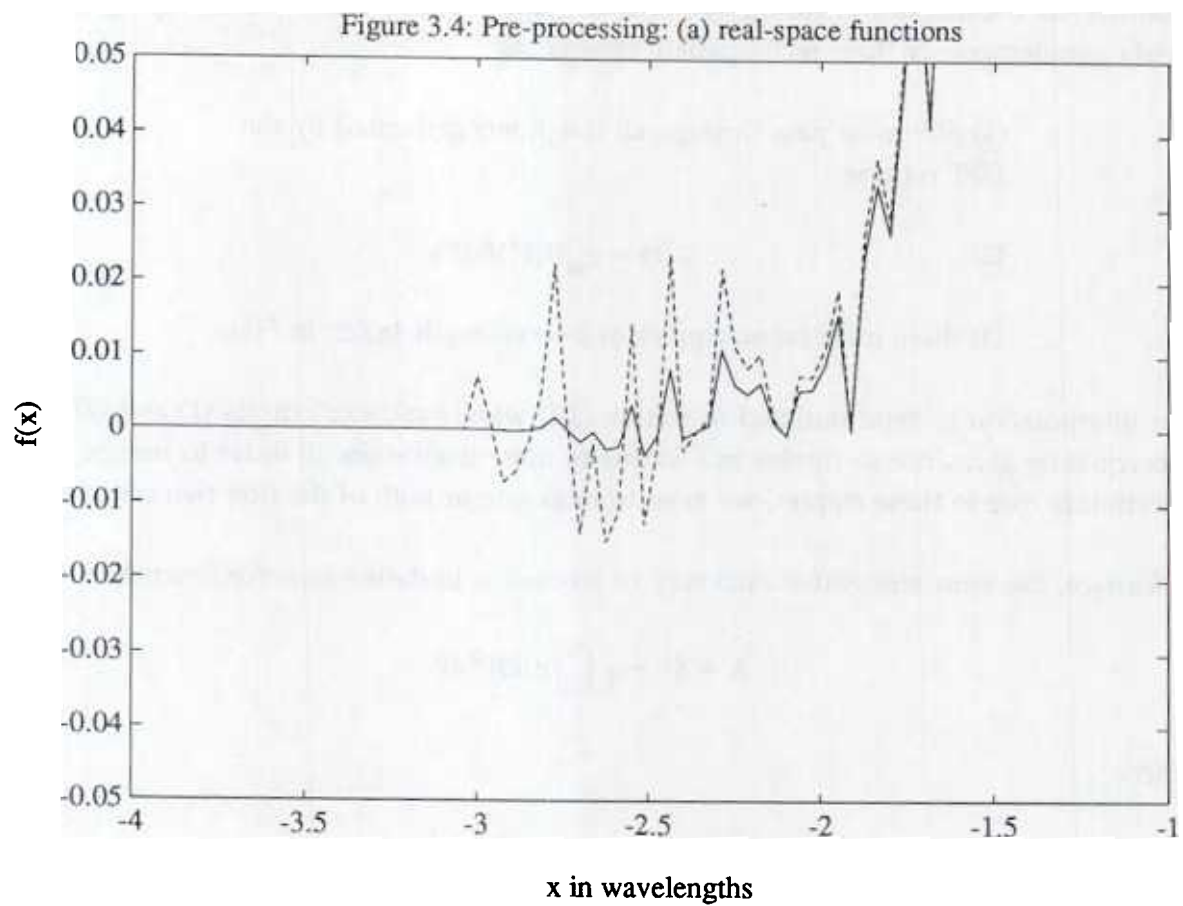
$$\varepsilon(k) = F(k) - \hat{s}_{\Delta x}(k, k') F(k'), \quad (3.3)$$

where δ^2 is the sum of the squares of the deviations of $F(k)$ from the DFT output values and where γ is a preset parameter. The next step is to minimise Δ by choosing the best values of the coefficients α_i in the expansion

$$F(k) = \sum_{i=1}^n \alpha_i \psi_i(k)$$

of $F(k)$ over the set of n basis functions $\{\psi_i(k)\}$. Our choice of γ must reflect the relative contributions of noise truncation and overall noise in the DFT output to the total uncertainty. We could of course get an arbitrarily good fit by allowing the set of basis functions $\{\psi_i(k)\}$ to become more and more complete; in the limit of large n , this would amount to the original interpolation procedure. The idea though is to eliminate the noise-truncation ripples by restricting the basis functions appropriately.

In real space, the third constraint corresponds to a requirement that $f(x)$ should not change sharply at the truncation points. As an alternative to the k -space procedure outlined above, therefore, we could pre-process $f(x)$ in real space by replacing the measured data with artificially generated numbers in the outer regions of the scan, where the signal is submerged in noise. The artificial points would be a simple extrapolation of the significant part of $f(x)$ down to a value of zero at the truncation point. The broken line in figure 3.4a, for example, shows the data of figure 3.3a (full line) on an enlarged scale, and with a (randomly) different noise component. The full line in figure 3.4a shows



exactly the same data, modified by being multiplied in the two noise-dominated regions by a factor of the form

$$\left(\frac{1}{2}\right)(1+\cos(ax+b))$$

with the as and bs chosen such that the factor is zero at the truncation point and rises over one half-cycle to 1 at the point where the signal starts to emerge from the noise.

Figure 3.4b shows the effect of this pre-processing on the resulting k -space function. The amplitude of the ripple in the original noisy, truncated function (broken line) is reduced by pre-processing (full line). This method of replacing the noise-dominated data points is very crude, however; we would probably get a larger reduction in the ripple by fitting the data points in the signal-dominated region, using a set of truncation-free functions, *e.g.*

$$(x-x_{trunc})^{2n} \quad (n=1,2,3\dots)$$

at each end of the scan, and then using the fitting parameters to construct artificial data points in the noise-dominated regions.

Pre-processing of this sort would go some way towards meeting criterion no 3 above (no truncation ripples in $F(k)$). The other two criteria would have to be met by a fitting process in k -space, similar to that already described. Again, the parameter γ would have to be chosen to reflect the relative importances of noise truncation and overall noise level.

The above discussion will be the basis for some further work on how best to obtain $F(k)$ from the DFT output.

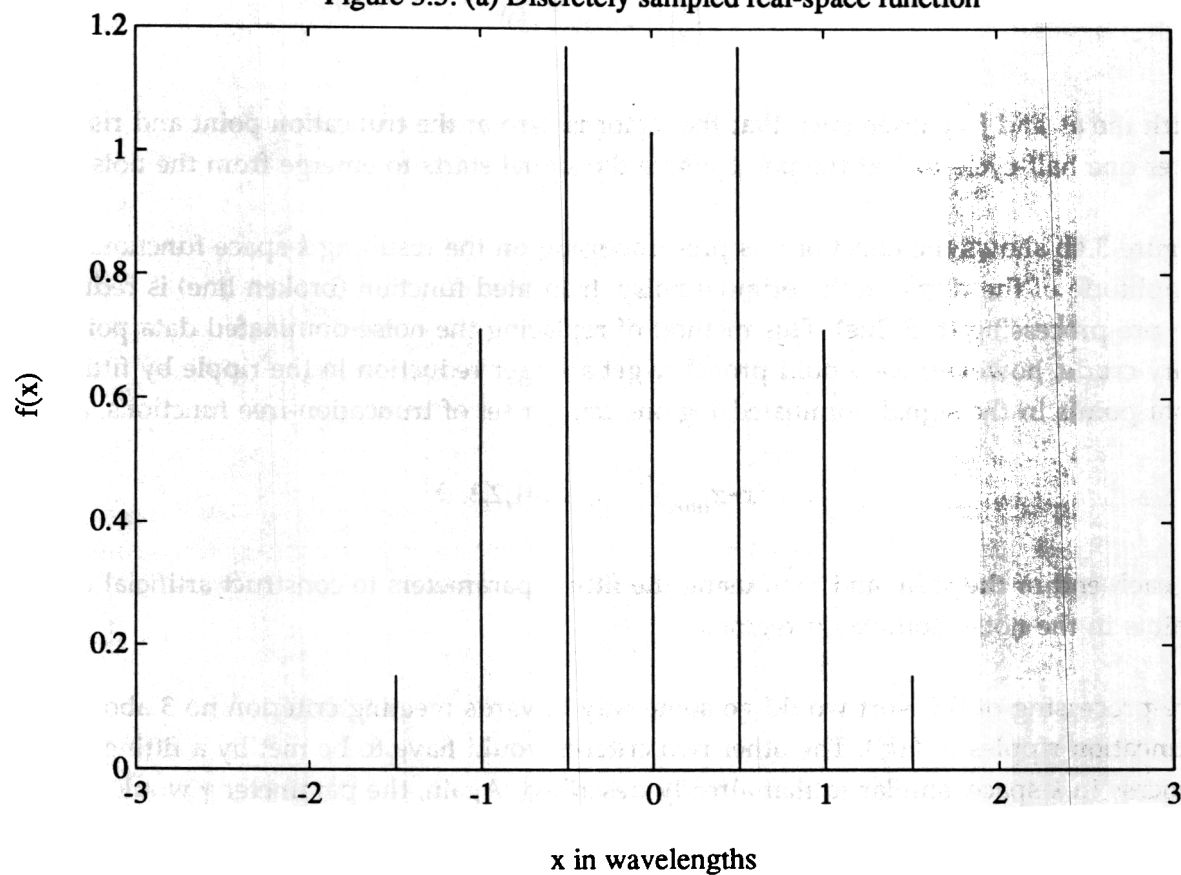
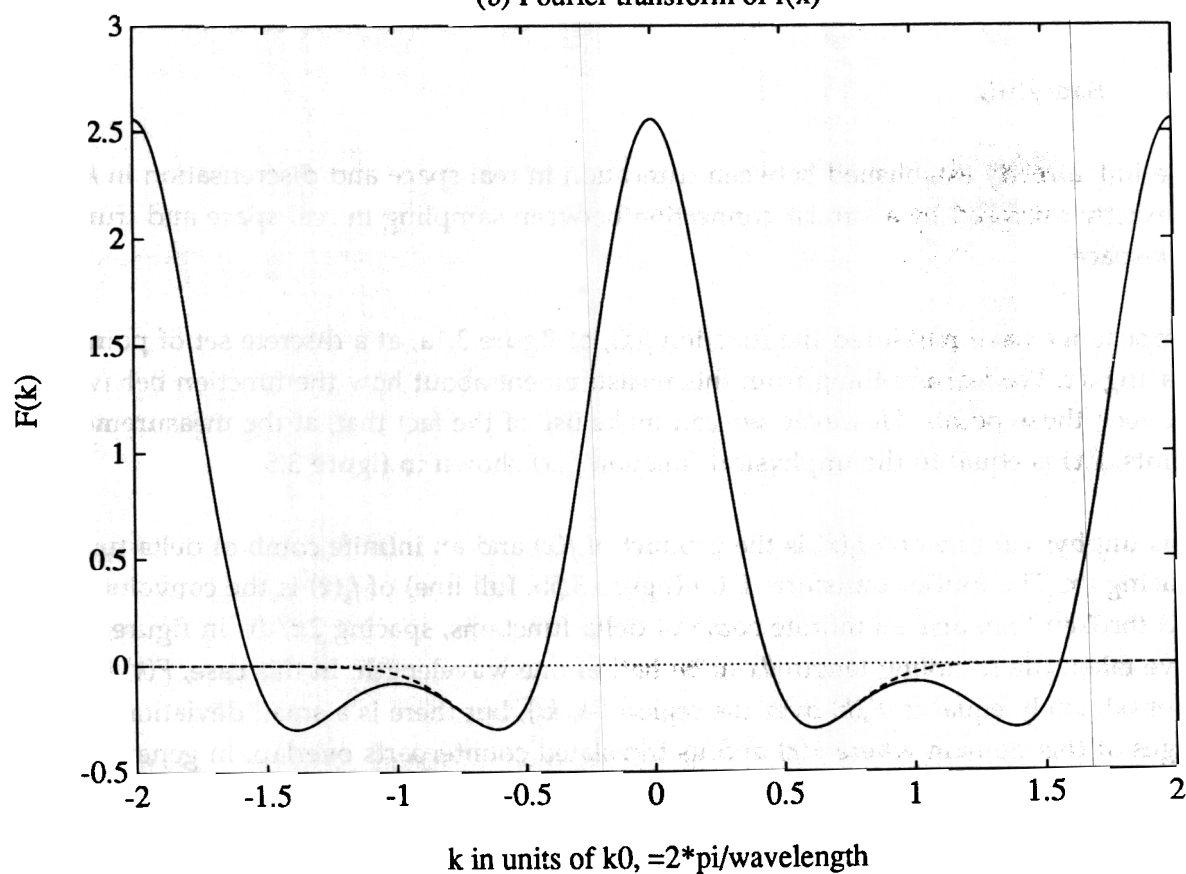
3.3 Sampling

The link already established between truncation in real space and discretisation in k -space is exactly mirrored by a similar connection between sampling in real space and truncation in k -space.

Suppose we have measured the function $f(x)$, of figure 3.1a, at a discrete set of points, spacing dx . We learn nothing from this measurement about how the function behaves between those points. However, we can make use of the fact that, at the measurement points, $f(x)$ is equal to the unphysical function $f_s(x)$ shown in figure 3.5.

This unphysical function $f_s(x)$ is the product of $f(x)$ and an infinite comb of delta functions, spacing dx . The fourier transform $F_s(k)$ (figure 3.5b, full line) of $f_s(x)$ is the convolution of $F(k)$ (broken line) and an infinite comb of delta functions, spacing $2\pi/dx$. In figure 3.5a, I have taken the sampling length dx to be half of one wavelength. In this case, $F(k)$ is approximately equal to $F_s(k)$ over the region $\{-k_0, k_0\}$, but there is a small deviation at the edges of this domain where $F(k)$ and its translated counterparts overlap. In general, the repeat distance Δk in k -space is given by

Figure 3.5: (a) Discretely sampled real-space function

(b) Fourier transform of $f(x)$ 

$$\Delta k = 2\pi / dx$$

The functions $f(x)$ and $f_s(x)$ are identical at all points on the comb, and their fourier transforms $F(k)$ and $F_s(k)$ are identical in $\{-\Delta k/2:\Delta k/2\}$, provided that $|F(k)|$ is zero outside this domain. On this condition, then, we can take the fourier transform of the discretised real-space function $f_s(x)$ as an estimate of $F(k)$ within $\{-\Delta k/2:\Delta k/2\}$.

To make use of this fact, though, we have to be sure that $F(k)$ is negligibly small outside $\{-\Delta k/2:\Delta k/2\}$. In the far field, we know that there can be no waves present with $|k| > k_0$; it follows that $|F(k)|$ is zero outside $\{-k_0:k_0\}$, and therefore that a sampling length of half of one wavelength is sufficiently small. In the near field, the presence of evanescent modes means that $F(k)$ may extend out beyond $\{-k_0:k_0\}$. The question: how closely do we need to sample, then leads us into circular arguments. The only way to find out is to use a small sampling length first, ascertain $F(k)$ over a wide domain, inspect $F(k)$ to see how far out in k -space it extends, and thereby estimate the corresponding maximum allowed value of dx . This constraint on dx is known as the Nyquist criterion, and can be summarised

$$dx \leq \frac{\lambda}{2}$$

Because our measured data set is both discretised and truncated, the resulting k -space functions are both band-limited and discretised. The band-width in k -space is inversely proportional to the real-space sampling interval and the step size in k -space is inversely proportional to the real-space scan size:

$$\begin{aligned}\Delta k &= 2\pi / dx \\ dk &= 2\pi / \Delta x\end{aligned}$$

The fast fourier transform (FFT) algorithm (see *e.g.* Oppenheim and Schaffer 1975) produces estimates of the values of $F(k)$ at n equally-spaced k -points. It is therefore very well suited to the processing of finite, discrete data sets.

Readily available FFT routines such as NAG C06FUF take the origin of the real-space data set (as in k -space) to be its bottom left-hand corner. We have to make use of the periodicity of $F_s(k)$, illustrated in figure 3.5b, to map the four quadrants of the FFT output data set into the required domain. This quadrant-by-quadrant mapping is shown in figure 3.6.

3.4 Probe correction

An ideal electric field probe at position r would produce a voltage proportional to one cartesian component of the electric field at r . Real field probes take an average of the field in a region centred on the point r , and the voltage V they produce can be written

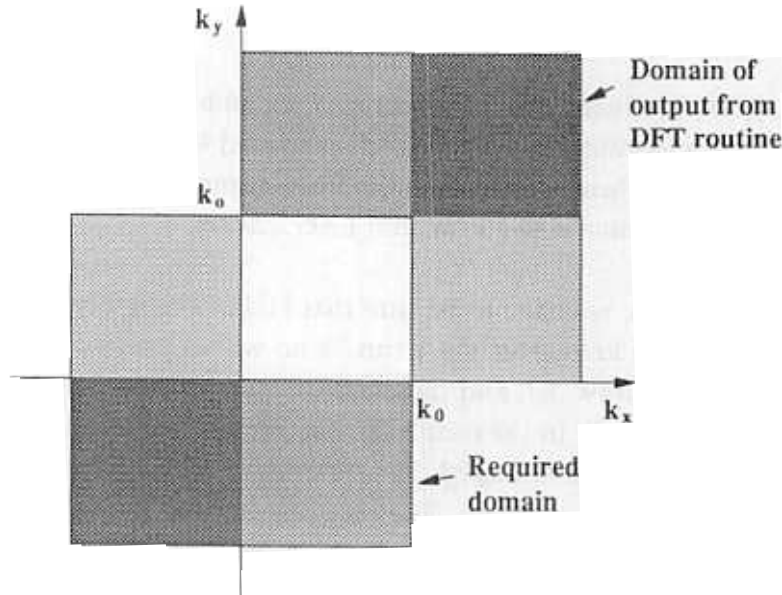


Figure 3.6

$$V(x, y, z) = \Psi \{ E(x', y', z') \} \quad (3.4)$$

where $E(x, y, z)$ is the field distribution in the absence of the probe. The probe operator Ψ acts on the complex vector E to give the complex scalar V . For an ideal probe, Ψ would be given by:

$$\Psi \equiv \int_{-\infty}^{\infty} dx' \delta(x-x') \int_{-\infty}^{\infty} dy' \delta(y-y') \int_{-\infty}^{\infty} dz' \delta(z-z') \mathbf{n}.$$

where $\mathbf{n}$ is the unit vector in the polarisation direction of the probe. We can see immediately from this special case of Ψ that it is impossible to invert uniquely equation (3.4) and calculate E , given V and Ψ , because there is not enough information in the scalar function V to obtain the three components of the vector function E . (In the language of linear algebra: if we expand V and the three components of E over a set of basis functions, the matrix P representing the operator Ψ has three times more rows than columns; the problem of generating a pseudo-inverse of P is underdetermined.)

Consider now the special case of an aperture-antenna probe measuring the electric field in the plane $z=z_m$ (figure 3.7). We can regard the voltage produced by the probe as the sum of contributions from all points in the aperture. These contributions will be weighted according to the path lengths between the point in question and the receiver. Let the weighting function be p ; then Ψ is given by

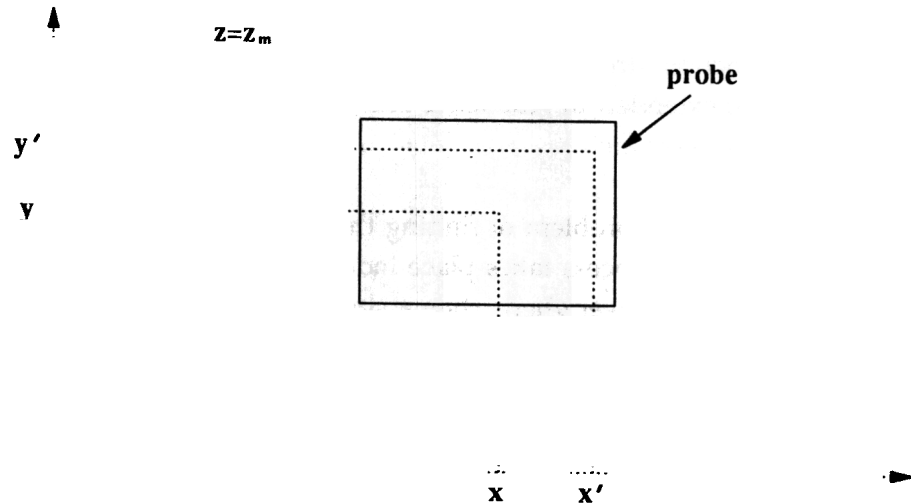


Figure 3.7

$$\Psi \equiv \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' p(x'-x, y'-y).$$

and V is therefore given by

$$V_{z_m}(x, y) = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' p(x'-x, y'-y) \cdot E(x', y', z_m)$$

Since (3.5) is the sum of three convolutions (one for each cartesian component of the vectors $\mathbf{p}$ and $\mathbf{E}$), it follows by the convolution theorem that the fourier transform V^k of $V(Z_m)$ is given by the scalar product

$$V^k(k_x, k_y) = p^k(k_x, k_y) \cdot E^k(k_x, k_y)$$

where the components of the two-vector V^k are the plane-wave spectra $s_x(k_x, k_y)$ and $s_y(k_x, k_y)$. These plane-wave spectra correspond to the $s(k_x, k_y)$ of section (2.2) above, with e defined as E_x, E_y respectively. It is these crosspolar and copolar plane wave spectra which we need in order to calculate the far field.

The function p^k is the fourier transform of the aperture weighting function p and this has to be determined, either by a separate characterisation of the probe or by *in-situ* near-field characterisation using a three-antenna technique. (A vanishingly small probe would give the ideal weighting function already mentioned.) Generally, as the probe aperture gets larger, p becomes more extended in real space and probe correction becomes correspondingly more important.

Given p^k , we are left with the problem of finding the two-vector E^k given the value of its scalar product with p^k . This process takes place independently at all points in k -space. If one component of p is zero everywhere, then a single scan, with the probe in one fixed orientation will suffice to determine one single component of E ; which component we get depends on which probe orientation we chose. This special case arises with a perfectly linearly polarised probe.

More generally, and more realistically if the method is to be applicable to horn probes, p^k will comprise two non-zero components, and to solve (3.6) for E^k we will need to perform two scans, with the probe in two different orientations. (Changing the probe orientation relative to the measurement coordinate system effectively changes p^k , giving us a second equation like (3.6), linearly independent of the first.)

3.5 Interaction

Perturbation is an inevitable consequence of attempting a measurement and any field probe, whatever its design, perturbs the field in which it is immersed. Ultimately, if we tried to make a probe which perturbs the field minimally, we would be limited by the uncertainty principle, which implies a trade-off between perturbation and accuracy. In practice, there is a similar trade-off between perturbation and sensitivity, but the limiting factor is noise in the detection system.

Perturbation implies scattering. The probe scatters some of the radiation incident on it and, in a similar way, the antenna under test, while behaving primarily as a transmitter, is also a passive reflector when seen from the probe's point of view. It follows from these two facts that the driving field experienced by the probe will inevitably be modified by radiation first scattered by the probe and then reflected back towards the probe by the antenna under test. This effect, which for brevity I will call interaction, is a special case of the more general interaction between the probe and its surroundings, whatever they may be.

If our probe were a long way from the antenna under test, as it would be in extrapolation measurements, multiple interaction between test antenna and probe would make only a small contribution to the overall uncertainty in the gain and we would therefore choose the probe for its high sensitivity, without regard for how much it perturbs the driving field. However, if we wish to make measurements close to the antenna under test, as in near-field scanning, then we face the following dilemma: the more sensitive the probe, the larger it has to be and therefore the more it will scatter, giving rise to a larger uncertainty due to interaction.

By reducing the size of the probe, we can bring it closer to the antenna and still maintain an acceptably low level of multiple interaction. For example, scanning a 20 dB-gain waveguide-16 horn antenna, Howes (1990) obtained an uncertainty due to interaction of ± 0.06 dB in the boresight gain by scanning with a small probe horn of gain 10 dB at a distance of 5λ from the aperture of the antenna under test. However, it is futile to try to reduce the size of the probe itself beyond the point where the probe support and the waveguide feed to which the probe is attached are the main causes of perturbation of the driving field (as they proved to be in Howes's measurements). In this limit, we turn to scattering techniques such as the optically modulated scatterer; this novel, waveguide-free probe is being developed here at NPL as an alternative to conventional aperture-antenna probes in near-field scanning.

Appendix: Matlab program for generating figure 3.1

```

% trunc.m
%
% a matlab program for generating a simulated one-dimensional
% near field and the fourier transform of this near field. The
% near field is the sum of gaussian modes, centred on the origin.
% Each mode  $G_n$  is normalised by an appropriate constant  $k_n$ ; the
% coefficients  $c_n$  of each mode are supplied by the user.
%
% This program also simulates the effect of truncating the near
% field, as happens in experimental measurements.
%
% Recommended procedure:
%
% (1) Set n (the number of data points) to 200,
%     set span to 10.0 wavelengths,
%     set  $g0hwhh$  to 0.5 wavelengths,
%     set  $xplotrelspan$  to 1.0,
%     set  $kplotreldeltak$  to n,
%     set  $c0=1$ ,  $c2=1.5$ ; this gets a realistic field.
%
% (2) Adjust  $c0$ ,  $c2$ , etc. to get required field variation.
%
% (3) Adjust  $g0hwhh$  such that the number of points spanning
%     the significant part of  $f(x)$  equals the number of points
%     spanning the significant part of  $F(k)$ . This will depend
%     on the chosen values of the  $c_n$ .
%
% (4) Choose values of  $xplotrelspan$ ,  $kplotreldeltak$  to remove
%     the insignificant outer parts of the two curves from the
%     graphs.
%
% (5) The user is now free to vary n, and get lower or higher
%     resolution in real space, without altering the features
%     of the graphs. To get more resolution in k-space, increase
%     span, increase n and  $kplotreldeltak$ , and decrease
%      $xplotrelspan$ , all by the same factor.
%
%
% span=150.0;
% total width of data in real space in wavelengths
%
%  $g0hwhh=0.75$ ;
%  $G0$  half-width at half height in wavelengths

```

```

%
%
n=4000;
% number of steps (arrays run from 1 to n+1)
% (n must be an even number)
%
truncrelcurve=2.0;
% width of scan window rel fwhh of G0
%
xplotrelspan=0.04;
% width of plot window in real space rel span
%
kplotreldeltak=1800;
% width of plot window in k-space rel deltak
%
c0=1;
c2=1.5;
% coefficients of G0, G1 in real-space field
%
%
a=(1/g0hwhh)^2;
deltax=span/n;
x=-span/2:deltax:span/2;
%
% Set truncation points in real-space:
tlower=n/2+1-fix(truncrelcurve*g0hwhh/deltax);
tupper=n/2+1+fix(truncrelcurve*g0hwhh/deltax);
%
% Set plotting window in real space:
xplower=n/2+1-fix(xplotrelspan*n/2);
xpupper=n/2+1+fix(xplotrelspan*n/2);
%
% Normalisation constants:
k0=(2.*a./pi).^0.25;
k2=(2.*a./(sqrt(3))).*k0;
%
% Gaussian mode functions:
g0=k0.*(exp(-a.*x.*x));
g2=k2.*x.*x.*(exp(-a.*x.*x));
%
% Real space function:
y=c0.*g0+c2.*g2;
%
% Truncate real-space function:
ytrunc(1:tlower-1)=0*y(1:tlower-1);
ytrunc(tlower:tupper)=y(tlower:tupper);
ytrunc(tupper+1:n+1)=0*y(tupper+1:n+1);

```

```

%
% Real-space graph:
plot(x(xplower:xpupper),ytrunc(xplower:xpupper),
x(xplower:xpupper),y(xplower:xpupper))
title('Figure 3.1: (a) Real-space functions');
xlabel('x in wavelengths');
ylabel('f(x)')
!copy trunc.m trunc.meta
!dl trunc.meta;*
meta trunc.meta
%
% Take discrete fourier transform:
y2=fft(y,n);
y1=real(fftshift(y2));
y2=span.*y1./n;
%
% Take DFT of truncated function:
y3=fft(ytrunc,n);
y4=real(fftshift(y3));
y3=span.*y4./n;
%
% Fix k-axis scale:
deltak=1/span;
k=-n*deltak/2:deltak:n*deltak/2;
kplower=n/2+1-fix(kplotreldeltak/2);
kpupper=n/2+1+fix(kplotreldeltak/2);
%
% correct for x-displacement of origin of data:
correct=0:1:n-1;
correct=cos(pi.*correct);
y3=y3.*correct;
y2=y2.*correct;
%
% K-space plot:
plot(k(kplower:kpupper),y3(kplower:kpupper),...
k(kplower:kpupper),y2(kplower:kpupper))
title('(b) K-space functions');
xlabel('k in units of k0, =2*pi/wavelength');
ylabel('F(k)')
meta trunc.meta

```

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