

Investigation of Non- Linearity in Long Fibre Amplifiers

Pete Tomlins

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Abstract

This report is intended to identify the significance of optical non-linear effects likely to occur in long Erbium Doped Fibre Amplifiers used for L-Band amplification. The work consists of both computational simulations and an experimental study.

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Approved on behalf of Managing Director, NPL
By Dr. Stuart Pollitt, Head of Centre for Electromagnetic and Time Metrology

Summary

This report summarises a programme of work to identify the significance of optical non-linear effects in fibre amplifiers employing long (> 100 metres) lengths of active fibre. The study concentrates on Erbium Doped Fibre Amplifiers designed for L-Band optical signal amplification.

The investigation finds evidence that non-linearities do occur to a significant magnitude within long fibre amplifiers. This result is consistent with results reported by component manufacturers. Therefore, it is suggested that further practical investigation is undertaken, considering the implications of optical non-linearities in amplifiers on loaded communications system.

During the implementation of a computer model of a fibre amplifier, it was found that there may also be the requirement for traceable measurement, and standardised measurement techniques for the fundamental parameters of erbium doped fibre. Such measurements would enable direct comparison of similarly specified doped fibres from different manufacturers.

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1. Introduction

Long fibre amplifiers, employing active fibre lengths of greater than 100 metres, have come about from the requirement to amplify optical signals over a broad spectrum of wavelengths beyond the capabilities of traditional fibre amplifiers. The primary drive in this region has come from the telecommunications sector where the push to increase network capacity has pointed towards the relatively low attenuation of standard optical transmission fibres across a range of wavelengths previously not exploited. It is found that wavelengths in the L-Band (1570 – 1611 nm)¹ can be amplified in a similar way to C-Band wavelengths (1530 – 1569 nm) with Erbium Doped Fibre Amplifiers (EDFAs). The most apparent difference in the amplifier designs for L-Band is the long lengths of Erbium Doped Fibre (EDF) required for comparable gain to traditional C-Band amplifiers. Therefore within long amplifiers we may find an ideal environment for detrimental optical non-linear effects to occur.

This project aims to determine whether optical non-linearities in long fibre amplifiers are of significant importance, and if so which measurements should be made to provide traceability. Current measurement service capabilities should be considered and an assessment made of the requirement for the development of these to support further work on non-linearity measurements in amplifiers. It is also intended that the use of computational modelling will give an insight into the fundamentals of EDFAs that will help us to understand where better metrology of EDF parameters is needed.

1.1. *Erbium Doped Fibre Amplifiers*

All optical fibre amplifiers work by transferring optical power from a ‘pump’ beam to another ‘signal’ beam of light within an optical fibre thus producing gain in the signal over the length of the amplifying fibre. In the case of the EDFA this transfer of energy from one beam to another occurs by stimulated emission of light in the same way as in a laser.

L-Band EDFAs require much longer lengths of EDF than their C-Band counterparts for comparable gain. This is due to the absorption and emission cross-sections of the Er^{3+} ion that are responsible for amplification. The probability of a photon of a given energy ϵ_1 being absorbed is proportional to an ions absorption cross-section σ_a , and likewise the probability of a photon of a given energy ϵ_2 being emitted is proportional to the emission cross-section σ_e of the ion at that energy. For pump wavelengths of 980 and 1480 nm the emission cross-section of the Er^{3+} ion is much lower than it’s absorption cross-section, however large emission cross-sections occur in the region defined by the C-Band, leading to high gain over this spectrum of wavelengths. Amplification can still occur in the L-Band in EDF but the emission cross-sections at L-Band wavelengths become significantly smaller leading to a much lower probability of stimulated emission, and hence lower gain. Therefore one way to increase the gain is by using longer lengths of EDF.

¹ ITU Grid Specification obtained from <http://www.bayspec.com/ITU-DWDM.htm>

It is immediately obvious that for accurate modelling and design of erbium doped fibre amplifiers it is essential to have a good measurement of the emission and absorption cross-sections at the pump and signal frequencies. These are dependent on the host material along with other dopants and therefore will differ between fibres. Other essential parameters in determining amplifier characteristics are the total density of erbium ions, the distribution of ions across the fibre and the fraction of the signal and pump that couple into this active region.

Assumptions about many of these values enable manufacturers to create useable computational models as design tools for fibre amplifiers, however models are of only limited use, and the final product must be tweaked by trial and error. A simplified model may be useful for estimating the magnitude of various non-linear effects in long lengths of EDF.

1.2. Optical Non-Linear Effects

Optical non-linear effects occur in the situation where the optical intensity within a dielectric becomes sufficiently high such that the motion of bound electrons becomes anharmonic in response to the applied electromagnetic field (*Appendix A*). In turn this causes changes to the fundamental optical properties of the dielectric, and hence alters the way in which light propagates through the material.

The way in which the electromagnetic field propagates through a dielectric is governed by the polarisation field $\mathbf{P}$, that is to say that the bound electrons act like oscillating dipoles in the presence of an applied field, with induced dipole moment $\mathbf{P}$ per unit volume.

In the linear regime the polarisation is written simply as being proportional to the applied field $\mathbf{E}$. [1]

$$\mathbf{P} = \epsilon_0 \chi \mathbf{E} \quad 1.1$$

However as the intensity of the applied field increases the dipole response becomes non-linear and the polarisation can be expressed in terms of a power series. [2]

$$\mathbf{P} = \epsilon_0 (\chi \cdot \mathbf{E} + \chi_2 : \mathbf{E}\mathbf{E} + \chi_3 : \mathbf{E}\mathbf{E}\mathbf{E} + \dots) \quad 1.2$$

Where χ_n is the nth order susceptibility. In optical fibres it is the third order susceptibility that is responsible for the lowest order non-linear effects. In many non-linear effects the contribution from χ_3 is manifested through changes to the refractive index which has a second order intensity dependent term discussed in section 2.1.1. Therefore, in order to characterise the magnitude of optical non-linearities the measurement of second order refractive index would be desirable. In a pre-built production amplifier this is not a straight forward measurement, and therefore we must look for alternative measurements of equal use.

Commonly studied non-linear effects in optical fibres are Self Phase Modulation (SPM), Cross Phase Modulation (XPM), Four Wave Mixing (FWM) and Stimulated Raman and Brillouin Scattering (SRS and SBS).

1.2.1. Self Phase Modulation (SPM)

SPM is a direct response to a change in refractive index due to high intensity by an optical pulse. The most prominent effect of SPM on a single pulse is the phenomena of spectral broadening. This refers to the way in which different intensities of a monochromatic pulse travel at different speeds due to the intensity dependence of the refractive index. This causes the leading edge of the pulse to have a longer wavelength, whilst the trailing edge bunches up to have a shorter wavelength. The net effect is a pulse made up of a range of frequencies beating together. Such broadening distorts the optical pulse, and is referred to as SPM induced chirp. The non-linear phase shift encountered can also lead to wave mixing through the satisfaction of what is known as the phase matching condition.

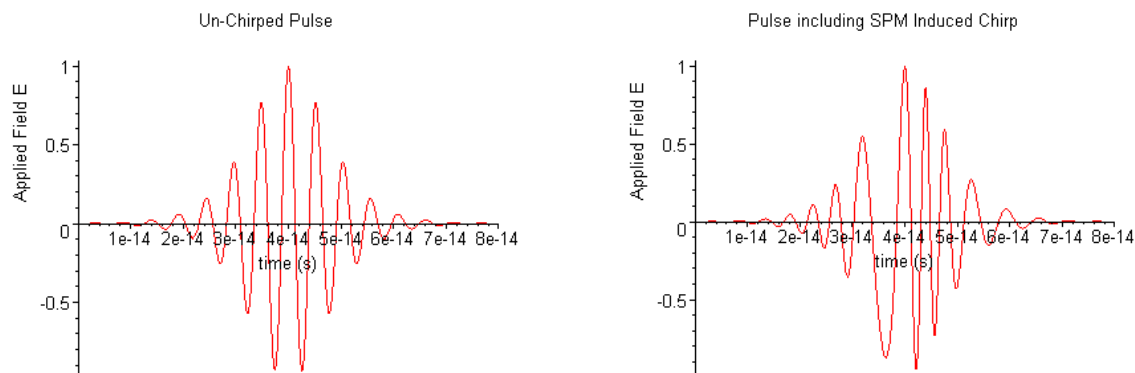


figure 1.1a. An 80 femtosecond un-chirped pulse carrying a 1550 nm wavelength signal. 1.1b. The same pulse spectrally broadened by SPM induced chirp.

SPM can also occur when two co-polarised spectrally overlapping beams beat together along an optical fibre. The envelope function looks like a train of pulses that can experience spectral broadening.

1.2.2. Cross Phase Modulation (XPM)

Cross-phase modulation is the result of the same physical principles as SPM. It is the effect of one pulse causing phase modulation in other overlapping pulses propagating simultaneously within the fibre. Each pulse is modified according to the aggregate intensity. Spectral broadening induced by XPM can be important for spectrally close signals where it can lead to spectral overlap between the channels. Immediately we see that consideration of this effect must be important for multi-channel systems.

1.2.3. Four Wave Mixing (FWM)

FWM is the process whereby signal waves interact producing new waves at shifted frequencies. Quantum mechanically we describe the effect as two photons annihilating one another creating two new photons at shifted frequencies that satisfy the conservation of net energy and momentum. For the effect to occur efficiently phase matching must be achieved. This condition refers to the requirement that the sum of the initial frequencies must be equal

to the sum of the mixing products, and that the overall momentum must be conserved. The phases of the propagating waves can be tailored such that these conditions are met. When the phase matching condition is achieved along a fibre it will have a contribution from the non-linear phase shift induced by SPM. In fibre we most commonly observe the degenerate form of FWM whereby for two signal waves ω_1 and ω_2 , two new frequencies are generated at $2\omega_1 - \omega_2$ and $2\omega_2 - \omega_1$. The effect is a direct consequence of the third order non-linear susceptibility χ_3 and is given further theoretical treatment in Appendix A.

1.2.4. Stimulated Brillouin Scattering (SBS)

SBS occurs in optical fibres as a result of electrostriction, due to the propagation of a high intensity optical field. Quantum mechanically it is described as the annihilation of a pump photon, and the spontaneous creation of an acoustic phonon and stokes photon. The Stokes photon is downshifted in frequency from the pump by an amount determined by the phonon energy. In silica optical fibres the Stokes shift is typically ~ 11 GHz.

In optical fibre, the Stokes wave can only exist in the backwards direction for a forward propagating signal. Classically this is understood by considering the optical intensity induced refractive index variation along a length of fibre. Such a variation in refractive index forms a travelling Bragg grating from which part of the signal is reflected back down the fibre, at a wavelength Doppler shifted from the pump.

1.2.5. Stimulated Raman Scattering (SRS)

SRS in optical fibres is similar to SBS in that it can be described by photon-phonon interactions, and the emission of a Stokes photon. However, rather than creating a grating as in SBS, the high intensity optical wave interacts with the molecular vibrational modes of the host material. The amorphous structure of silica results in a broad vibrational spectrum. As a result the Stokes emission is also broadband and can be used for broadband optical amplification. Raman Stokes waves can propagate in both the forwards and backwards directions.

The Raman gain peak for silica optical fibre is downshifted from the pump frequency by ~ 13 THz, and has a gain profile width ~ 40 THz wide (*figure 1.2.*).

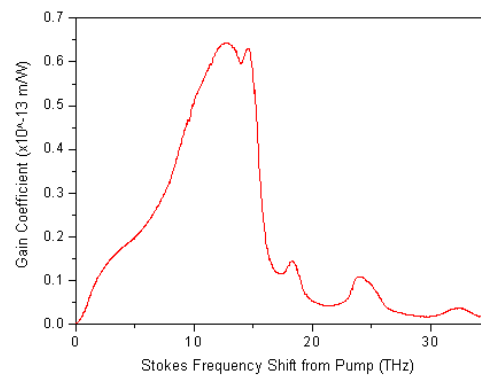


figure 1.2. Raman gain profile in pure Silica optical fibre [3]

2. Non-Linear Effects in Fibre Amplifiers

2.1. Scope

2.1.1. Studied Phenomena

With the development of L-Band fibre amplifiers, there has been a notable level of work carried out investigating their contributions to optical non-linear effects. In the published literature the topic has been approached in a variety of ways, covering both the beneficial and detrimental effects of non-linearity on communications systems [4,5,6]. It should be noted that conflicting results have been quoted regarding the impact of long amplifier induced optical non-linearity on communications systems.

2.1.2. Definitions

Fibre amplifiers are intrinsically non-linear in their behaviour since the amplified output signal does not follow a linearly proportional relationship to the input. However, it is not this ‘amplifier’ non-linearity that is considered here. Instead we focus exclusively on those effects mentioned in section 1.2 that are a direct consequence of a high intensity optical field propagating in the host material. These are effects that are not specific to fibre amplifiers but are intensity dependent and occur in pumped EDF, un-pumped EDF and standard single mode fibre alike [7,8]

2.2. Characterisation of Non-Linearity

2.2.1. Non-Linear Refractive Index n_2

The refractive index n of the dielectric is a function of the polarisation[9], and in the linear regime can be described by equation 2.1.

$$n = \sqrt{1 + \frac{P}{\epsilon_0 E}} \quad 2.1$$

Therefore, as the polarisation becomes non-linear due to higher order susceptibilities then so too does the refractive index.

The non-linear, or second order, refractive index n_2 is expressed as part of an intensity dependent term for the net refractive index $n(I)$ and has units of m^2/W .

$$n(I) = n_0 + n_2 I \quad 2.2$$

In general n_2 is small compared to n_0 , however at high intensities ($\sim 200 \text{ MW m}^{-2}$) such as those encountered within the small optical fibre core the term in n_2 can have an appreciable effect on the net refractive index. Non-linear effects caused by this change in refractive index are referred to as Kerr effects.

2.2.2. Effective Area A_{eff}

The effective area A_{eff} refers to the cross-sectional area over which the power distribution of the fundamental mode can be considered uniform, and is defined by:

$$A_{eff} = \frac{2\pi \left(\int_0^\infty |E_a(r)|^2 r dr \right)^2}{\int_0^\infty |E_a(r)|^4 r dr} = \frac{2\pi \left(\int_0^\infty I(r) r dr \right)^2}{\int_0^\infty I^2(r) r dr} \quad 2.3$$

Where $E_a(r)$ is a function of electric field with radius r .

If the fundamental mode distribution can be approximated to a Gaussian, as is often the case for standard single mode fibre with a signal at 1550 nm, then from 2.3 we can show (*Appendix A*) that A_{eff} becomes:

$$A_{eff} = \pi w^2 \quad 2.4$$

For a Gaussian field distribution $2w$ is the Petermann-II mode-field diameter defined for the near field by [10].

$$MFD = 2 \sqrt{\frac{2 \int_0^\infty |E_a(r)|^2 r dr}{\int_0^\infty \left[\frac{d}{dr} E_a(r) \right]^2 r dr}} = 2w \quad 2.5$$

The measurement of the mode-field diameter and effective area of single mode fibres are established services at NPL under the National Measurement System (NMS).

2.2.3. Non-Linear Coefficient γ

The non-linear coefficient is a useful parameter since from it we can determine the impact of second order refractive index dependent non-linearities on a system.

$$\gamma = \frac{n_2 \omega}{c A_{eff}} = \frac{2\pi}{\lambda} \frac{n_2}{A_{eff}} \quad 2.6$$

From (2.6) it is evident that if the non-linear coefficient γ can be determined, then from that we may obtain the second order refractive index.

It should be noted that γ has also been quoted as [11,12]

$$\gamma = \frac{n_2}{A_{eff}} \quad 2.7$$

For the remainder of this report non-linear coefficient γ shall refer to equation 2.7 since it has the easily understood units W^{-1} and does not contain wavelength.

The importance of the non-linearity becomes immediately obvious from the non-linear propagation equation used to study non-linear effects mathematically (*Appendix A*).

$$i \frac{\partial A}{\partial z} = -\frac{i}{2} \alpha A + \frac{1}{2} \beta_2 \frac{\partial^2 A}{\partial T^2} - \frac{2\pi}{\lambda} \gamma |A|^2 A \quad 2.8$$

Where A is the slowly varying amplitude of the electric field, α is the fibre attenuation, β accounts for pulse broadening due to the wavelength dependence of the refractive index. The last term in γ is responsible for non-linear effects such as phase-modulation.

When considering Self Phase Modulation, normalisation of eq. 2.8 and setting $\beta = 0$ (zero dispersion) results in the solution

$$U(z, T) = U(0, T) \exp[i\varphi_{SPM}(z, T)] \quad 2.9$$

where $U(0, T)$ is the normalised field amplitude at $z = 0$ and φ_{SPM} is the non-linear phase shift given by

$$\varphi_{SPM}(z, T) = \gamma \frac{2\pi}{\lambda} |U(0, T)|^2 z_{eff} P_0 \quad 2.10$$

for effective distance z_{eff} and peak pulse power P_0 . We see clearly that the coefficient of non-linearity plays an important role in determining by how much the phase shifts due to SPM. SPM induced phase shift is also responsible for spectral broadening of optical pulses. The non-linear phase shift contributes to the phase matching condition required for Four-Wave Mixing and hence the FWM efficiency oscillates along the fibre length.

The non-linear phase shift φ_{SPM} can be measured by generating a high intensity beat signal using two co-polarised sources of similar wavelength. This technique can be regarded as either partially degenerate FWM (*Appendix A.iv*) or the result of phase modulation in the beat frequency giving rise to extra frequencies either side of the source frequencies (*Appendix A.i*).

In the case of Cross-Phase Modulation XPM, the non-linear phase shift of the j th signal is given for two signals with non-overlapping spectra as

$$\varphi_j = \gamma_j \frac{2\pi}{\lambda_j} (P_j + 2P_{3-j})L \quad 2.11$$

indicating that XPM is twice as efficient as SPM for discrete spectra.

2.2.4. Effective Length

Effective length refers to the length of zero-attenuation fibre in which the same magnitude of non-linearity would be observed as in a real piece of fibre length L with attenuation α . For non-amplifying fibre including EDF without pumping, effective length L_{eff} is expressed as follows:

$$L_{eff} = \frac{(\exp(-\alpha L) - 1)}{\alpha} \quad 2.12$$

For lossy fibre it can be seen that the effective length will be shorter than the actual length of fibre. However, in a fibre amplifier we replace attenuation with gain, and hence find the effective length to be longer than the real length of amplifying fibre.

In general the effective length can be expressed as an integral of the optical power along the length of the fibre [13].

$$L_{eff} = \frac{1}{P_{in}} \int_0^{L_{EDF}} P(z) dz \quad 2.13$$

This may be useful when considering amplifiers since the gain will be a complex function of position along the EDF. It is possible to estimate L_{eff} by means of numerical simulation of an Erbium doped fibre amplifier displaying similar characteristics to those observed in a practical experiment, although perhaps further work is required to verify the validity of this approach.

2.3. Measurement Techniques

2.3.1. Partially Degenerate FWM/ SPM Approach

To characterise the magnitude of Kerr type optical non-linearities in fibre amplifiers the non-linear coefficient has been identified as a useful parameter to measure.

As previously discussed, the non-linear coefficient can be derived from the SPM induced non-linear phase shift ϕ_{SPM} . This quantity can be measured directly as a result of beat frequency wave-mixing in a length of optical fibre from two co-polarised source wavelengths. The observed degenerate four-wave mixing generates side-lobes either side of the source wavelengths. These can also be considered the result of SPM induced beat frequencies [14]. The non-linear phase shift is determined numerically from the ratio of the fundamental and first order side-band intensities, I_0 and I_1 respectively (*figure 2.1*).

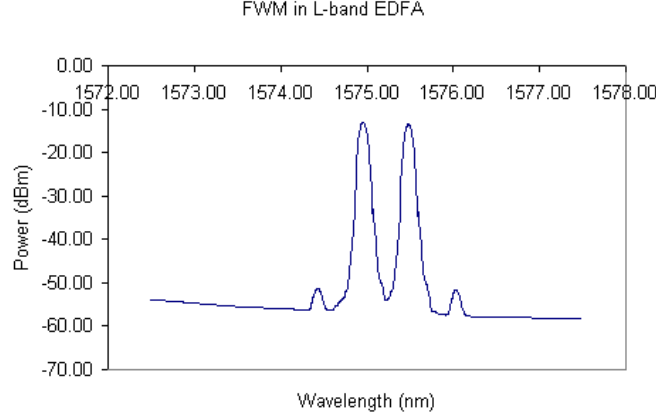


figure 2.1 Four-wave mixing in an L-Band fibre amplifier. The two fundamental frequencies at 1574.5 nm and 1575.5 nm with first order side-bands.

The mathematical relationship between $\frac{I_0}{I_1}$ and φ_{SPM} is given [11] as:

$$\frac{I_0}{I_1} = \frac{J_0^2(\varphi_{SPM}/2) + J_1^2(\varphi_{SPM}/2)}{J_1^2(\varphi_{SPM}/2) + J_2^2(\varphi_{SPM}/2)} \quad 2.14$$

Our numerical solution to this equation involves calculating intensity ratios for values of non-linear phase-shift from very nearly zero to 2π radians. Then we find the two calculated ratios in between which our measured intensity ratio falls. Taking φ_{SPM} to be along an imaginary x -axis and $\frac{I_0}{I_1}$ to be along an imaginary y -axis, the gradient between the two calculated intensity ratios is used to make a better guess at the non-linear phase shift. The corresponding intensity ratio is then compared to the measured ratio. This process is iterated between a decreasing phase shift separation until our ‘guess’ at φ_{SPM} yields a value equal to the measured intensity ratio from eq. 2.14.

Once a suitable value for the non-linear phase shift has been obtained we use the relation [12]

$$\varphi_{SPM} = \frac{4\pi}{\lambda} \gamma L_{eff} \bar{P} \quad 2.15$$

to calculate the non-linear coefficient γ . $\bar{P}$ is the net output power, in amplifiers we assume that significant FWM only occurs once the signal has reached its output power. 2.15 is similar to equation 2.10 but contains a multiple of 2 to account for all polarisation states.

Using this method to obtain the non-linear coefficient or n_2 requires that a number of other parameters be measured. These include the fibre effective length (eq. 2.15) and effective area (eq. 2.7). These are readily measured in bare fibre, however, in an ‘off the shelf’ amplifier, access to the active fibre is either not feasible or not possible. In this case an estimate of likely values for these parameters can be made through a combination of measurement, computer simulation and using values found in related literature.

Despite these drawbacks the chosen technique is advantageous since it requires the direct observation of the results of optical non-linearity. Therefore, without making any numerical analysis, the presence of non-linear behaviour can be observed and confirmed. Further to this, the observation of new frequency generation immediately confirms the presence of both FWM and SPM.

2.3.2. Computational Estimation

Computer simulations and mathematical modelling are often used to make predictions about physical systems. These simulations are then validated by experimental results. However, in the case where not all of the physical parameters that characterise a system can be measured a theoretical model can be used to test values suggested in the literature.

Under this project computer models of FWM and SPM evolution along a length fibre, and an EDFA have been developed. These have been used to make predictions about the magnitude of non-linear effects in fibre amplifiers. Where appropriate results have been compared with the amplifier under test. This has been the approach of many publications on the subject [4-8].

The EDFA model that we use is based on a reducing the energy levels of the Er^{3+} ion to a two-level system. This is a well established approach, that with certain assumptions has been solved analytically [15]. This solution ignores Amplified Spontaneous Emission (ASE) and is therefore only valid for gains of around 20 dB, where the ASE is small. An analytical treatment of ASE in the two-level system can be found in [16].

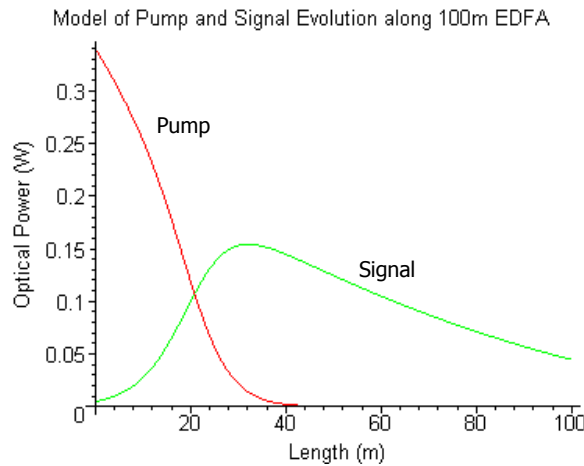


figure 2.2 Evolution of pump and signal along the length of an EDFA based on computational simulation of a two-level system.

The model is governed by two coupled differential equations for a single cw pump and signal propagating in the same direction along the EDFA, from $z=0$ to $z=L$.

$$\frac{dP_p(z)}{dz} = (N_2\sigma_p^{(e)} - N_1\sigma_p^{(a)})\Gamma_p P_p - \alpha_p P_p \quad 2.16$$

$$\frac{dP_s(z)}{dz} = (N_2\sigma_s^{(e)} - N_1\sigma_s^{(a)})\Gamma_s P_s - \alpha_s P_s \quad 2.17$$

where the excited state population N_2 is given by

$$N_2 = \frac{\frac{\tau\sigma_s^a}{Ah\nu_s}\Gamma_s P_s + \frac{\tau\sigma_p^a}{Ah\nu_p}\Gamma_p P_p}{\frac{\tau(\sigma_s^a + \sigma_s^e)}{Ah\nu_s}\Gamma_s P_s + \frac{\tau(\sigma_p^a + \sigma_p^e)}{Ah\nu_p}\Gamma_p P_p + 1} N. \quad 2.18$$

We take fibre attenuation into account by the term $\alpha_{p,s}$ for the pump and signal respectively, Γ is the overlap factor, defining the proportion of a beam propagating in the doped part of the fibre. τ is the spontaneous emission lifetime and σ is the Erbium ion emission and absorption cross-section at either pump or signal frequencies. These equations are obtained by assuming steady state conditions where the field and state populations N_1 and N_2 are independent of time.

In the formulation of a computer model many assumptions have been made about the values of each parameter. This is interesting since there seems to be little indication of what sensible values might be, and therefore often a trial and error approach is adopted.

Four-Wave Mixing is modelled for a standard transmission fibre using the approach of Shibata *et al* [17]. In this method the output power of the FWM generated light is given by

$$P_{FWM}(z) = \frac{1024\pi^6}{n^4 \lambda^2 c^2} D^2 \chi^2 \frac{P_i P_j P_k}{A_{eff}^2} e^{-\alpha z} \frac{(1 - e^{-\alpha z})^2}{\alpha^2} \eta. \quad 2.19$$

where P_{ijk} is the power of the i, j or k^{th} input signal, D is the degeneracy and is either 1, 3 or 6 depending on whether 3, 2 or 1 of the input signals is the same. χ is the third order non-linear susceptibility of the waveguide material, n is its refractive index and η is the FWM efficiency that is dependent on the phase-mismatch described in Appendix A. Degenerate FWM is considered here, where $i = j$.

An example of the FWM generated light evolution along a 25 km length of fibre is shown in *figure 2.3*.

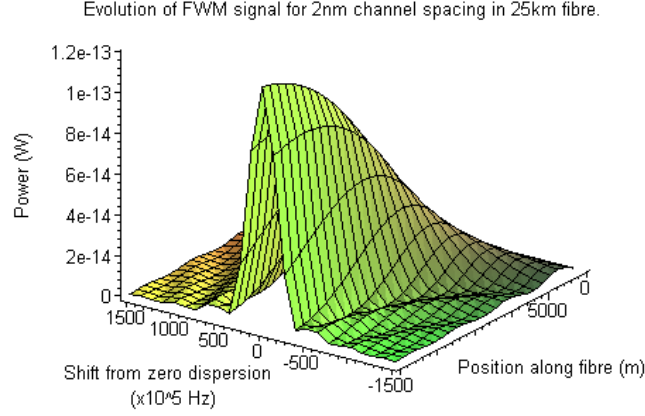


figure 2.3 Power Evolution of FWM light along 25 km of standard single mode fibre as a function of shift from the zero dispersion frequency and fibre length. This plot is the result of degenerate FWM where two signals at 2 nm spacing are launched into the fibre.

It can be more useful to consider the FWM efficiency along the fibre length that is a function of the phase-mismatch. The SPM induced phase shift becomes important at high intensities and affects the phase-mismatch. This is described in the last term in the phase-mismatch equation:

$$\Delta\beta = -\frac{\lambda^4 \pi}{c^2} \frac{dD_c}{d\lambda} 2(f_i - f_k)^2 (f_i - f_0) - \phi_{SPM} . \quad 2.20$$

$\frac{dD_c}{d\lambda}$ is the slope of the dispersion curve and $f_{i,k,0}$ are the input frequencies (i and k) and zero dispersion frequency (0).

The FWM efficiency is given by

$$\eta = \frac{\alpha^2}{\alpha^2 + (\Delta\beta)^2} \left[1 + \frac{4e^{-\alpha L} \sin^2(\Delta\beta L / 2)}{\{1 - \exp(-\alpha L)\}^2} \right] . \quad 2.21$$

Efficiency of FWM for two signals spaced from 2 to 10nm in 25 km fibre.

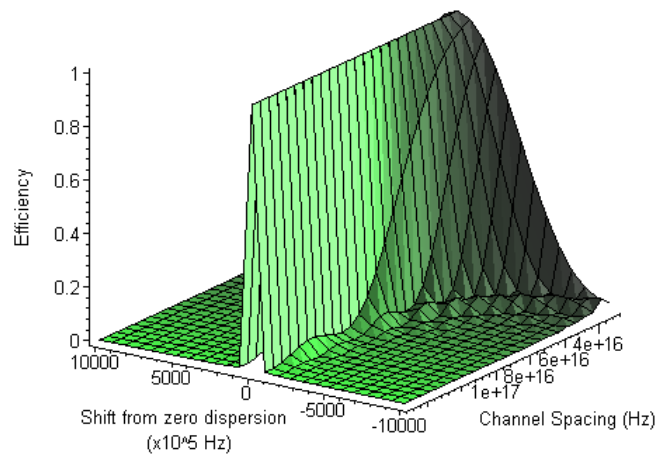


figure 2.4 FWM Efficiency for 25 km fibre for two channels spaced from 2 nm to 10 nm. The Efficiency becomes less broad with increasing channel separation.

To combine these models to predict non-linearity in an L-Band amplifier, the attenuation coefficient of the fibre (α) in equations 2.19 to 2.21, is replaced with the predicted amplifier gain. A result from this combined model is given in *figure 2.5*.

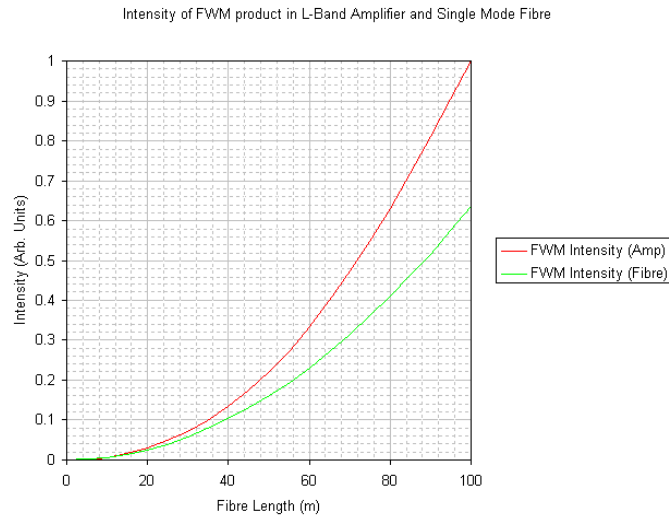


figure 2.5 The plot shows the intensity of a FWM product in an L-Band EDFA and an equivalent length of standard single mode fibre.

The plot shows that long lengths of pumped EDF are more efficient at generating FWM products than equivalent lengths of standard single mode fibre, even away from the fibre zero-dispersion, where FWM is most efficient.

3. Experimental Investigation

The practical experiments have been carried out using a JDS Uniphase OAB1592+20FA0 L-Band amplifier that has been characterised at NPL.

The gain profile is shown in *figure 3.1*, and reveals a region of maximum gain from 1555 nm to 1575 nm. The amplifier exhibits a reasonably constant and low noise figure across the L-Band.

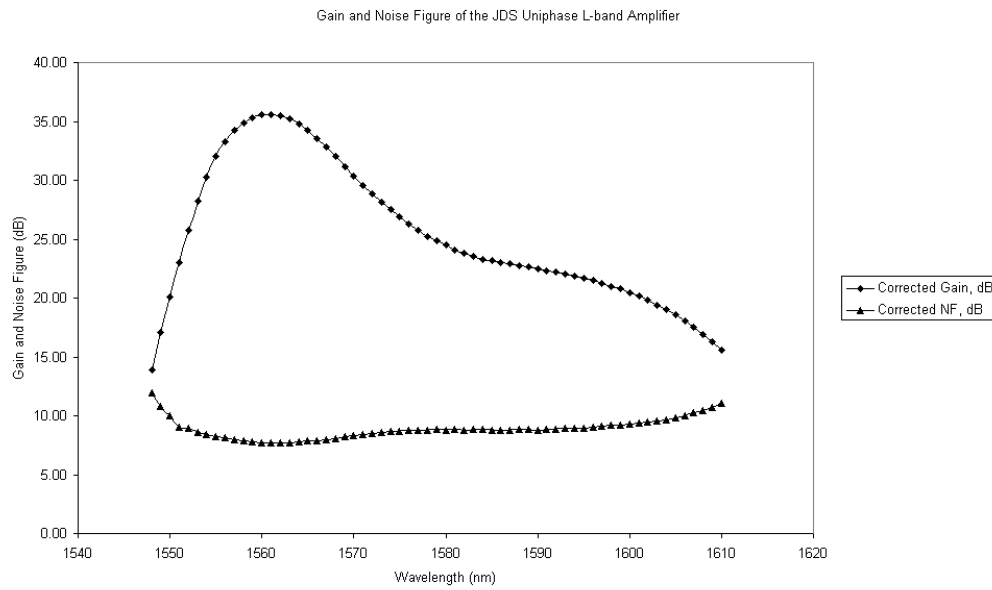


figure 3.1 Gain and noise figures for the JDS Uniphase L-Band amplifier used throughout this project for the investigation of non-linear effects in long amplifiers.

The length of EDF within the fibre amplifier was measured at NPL, using the NMS standard fibre length measurement service at a wavelength of 1618 nm. This wavelength was chosen so that a pulse would be transmitted with the EDFA switched off, therefore avoiding damage to the measurement equipment from amplified peak pulse powers.

The total length of fibre within the L-Band amplifier was found to be 116 m. This figure is not exclusive to just the EDF, but includes all of the fibre used to connect isolators, connectors and other parts inside the amplifier. We will assume that the EDF is the largest proportion of this length, ~100 m. Using the EDFA simulation software developed as part of this project, the active fibre effective length has been estimated from equation 2.13. This estimation technique yields an effective length, $L_{eff} = 226$ m. This value is used for all subsequent calculations.

3.1. Measurement of Non-Linear Coefficient in L-Band EDFA

The non-linear coefficient of the L-Band EDFA has been measured by the Four-Wave Mixing technique covered in section 2.3.1 using the apparatus shown in figure 3.1. The two laser diode sources were modulated at 16 MHz to suppress the effects of SBS.

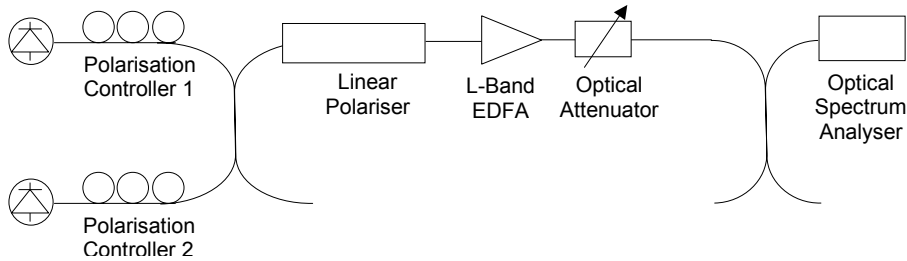


figure 3.2 Experimental apparatus configuration for measurement of non-linear coefficient in an L-Band fibre amplifier at 1575 nm by Four-Wave Mixing.

An initial experiment was undertaken to assess the magnitude of the Four-Wave Mixing effect in our L-Band amplifier. Measurements were made at a nominal wavelength of 1575 nm (L-Band) with a 1 nm channel spacing between input signals, using the configuration in figure 3.2. The experiment revealed that appreciable Four-Wave Mixing does occur within the amplifier under test, plotted results are shown in figure 3.3.

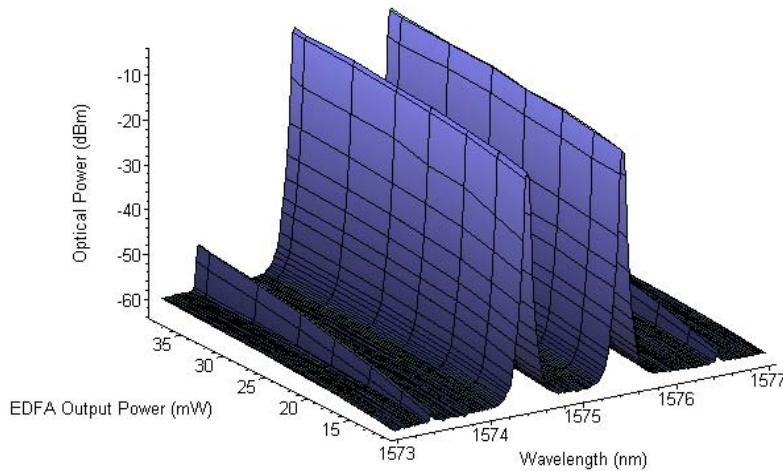


figure 3.3 Evolution of FWM with increasing gain from the L-Band EDFA. The plot shows the two signal wavelengths at 1574.5 nm and 1575.5 nm, and side-band mixing products at 1573.5 and 1576.5 nm.

From these results the non-linear phase shift for each amplifier input power is calculated to obtain the chart in figure 3.4.

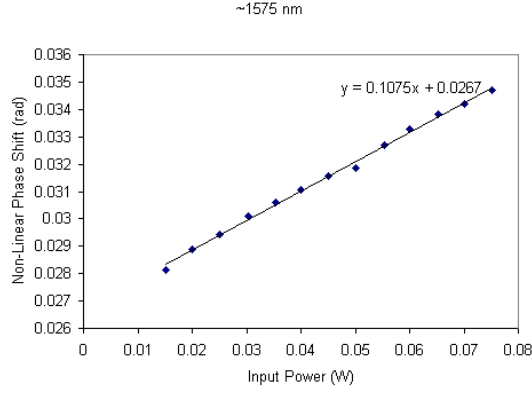


figure 3.4 Experimental results for the measurement of the non-linear phase shift in our L-Band amplifier at 1575 nm. The non-linear phase shift is calculated from the measured intensity ratios between the fundamental peaks and first order sidebands.

The non-linear phase shift is a result of self-phase modulation within the long length of EDF that makes up the L-Band amplifier. Theoretically it follows a linear relationship with the input power, which is confirmed by our experiment. This indicates that although the amplifier is characteristically non-linear in its amplification this causes little or no detrimental effect to our measurement of the non-linear phase shift, and that the contribution due to dispersion is minimal.

From the initial data and using equation 2.15 re-arranged for γ we obtain an estimated value of $\gamma_{EDF} = 1.20 \times 10^{-10} \text{ W}^{-1}$. This is of comparable magnitude with our own measurements of non-linear coefficient in Dispersion Shifted Fibre (DSF), where $\gamma_{DSF} \approx 3 \times 10^{-10} \text{ W}^{-1}$. However our initial non-linear coefficient measurement is low when compared with values quoted in reference [11] for DSF ($\gamma_{DSF} \approx 4 \times 10^{-10} \text{ W}^{-1}$), and reference [6] for L-Band EDF ($\gamma_{EDF} \approx 4.6 \times 10^{-10} \text{ W}^{-1}$).

From further measurements of the fundamental and sideband intensities at different wavelengths from the C-Band into the L-Band we obtain values represented in figure 3.5.

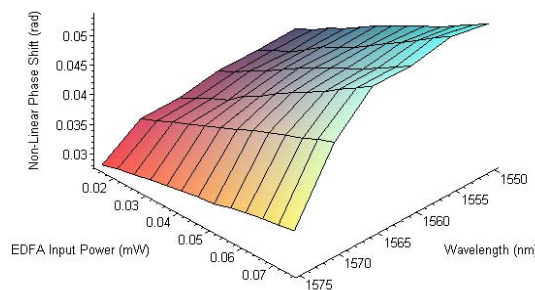


figure 3.5 Experimental results for the measurement of the non-linear phase shift in an L-Band amplifier at a range of wavelengths from 1550 nm to 1575 nm. The results show good linearity at each wavelength.

It is evident that the gradient changes as the wavelength moves into the L-Band, as too does the phase offset at zero power. The resulting non-linear coefficient values compare with those measured for DSF as follows:

λ (nm)	1550	1555	1560	1565	1570	1575
$\gamma_{EDF} (\times 10^{-10} \text{ W}^{-1})$	3.57	3.48	3.24	3.46	2.48	1.16
$\gamma_{DSF} (\times 10^{-10} \text{ W}^{-1})$	3.31	-	3.69	-	3.58	3.42

In order to obtain a high enough input signal to saturate the L-Band amplifier we place a C-Band amplifier between the linear polariser and the L-Band amplifier. When we represent graphically the gain curve of a similar C-Band amplifier to ours, and compare this with the measured L-Band amplifier non-linear coefficient we find a relationship between them, *figure 3.6*.

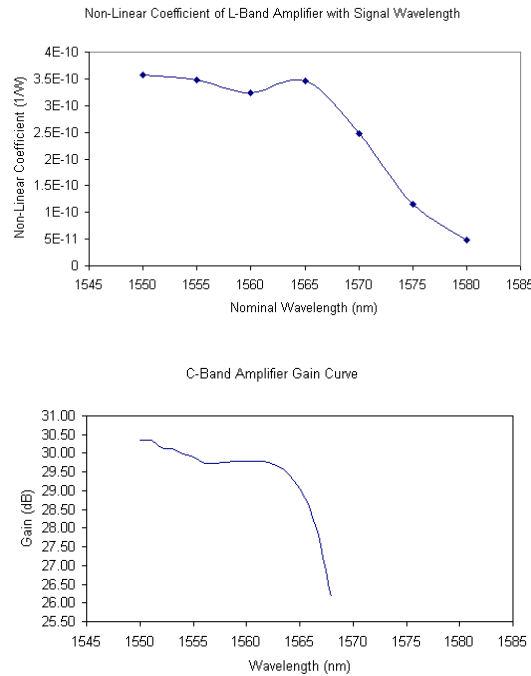


figure 3.6 Comparison between the C-Band ‘boost’ amplifier gain curve and measured non-linear coefficient within the L-Band amplifier show good correlation.

If we attribute the change in non-linear coefficient to the gain characteristics of the C-Band boost amplifier this suggests that with this arrangement only measurements made within the C-Band range of wavelengths are a good indication of the non-linear coefficient. It turns out that values for wavelengths 1550 nm to 1565 nm are in reasonable agreement with those quoted for DSF and EDF in the literature, appearing to read $\sim 10\%$ low.

The non-linear phase-shift is a good indication of the magnitude of phase modulation effects. From it, the spectral broadening due to SPM induced chirp can be quantified. For example for a pulse of temporal width $T_0 = 0.1$ ps the maximum spectral broadening is given by equation 3.1, from the maximum non-linear phase shift, $\phi_{SPM}^{\max}$.

$$\delta\omega_{\max} = 0.86 \frac{1}{T_0} \varphi_{SPM}^{\max} \quad 3.1$$

For two propagating cw signal beams the pulse width can be expressed as the time period of the envelope function with angular frequency $\Delta\omega = \omega_1 - \omega_2$ where $\omega_1 > \omega_2$. Therefore 3.1 becomes

$$\begin{aligned} \delta\omega_{\max} &= 0.86 \Delta\omega \varphi_{SPM}^{\max} \\ \therefore \\ \delta\omega_{\max} &= \frac{1.72\pi\varphi_{SPM}^{\max}}{\Delta\lambda}. \end{aligned} \quad 3.2$$

From these equations it is now possible for us to estimate the magnitude of spectral broadening within our amplifier at different powers, and make comparisons with DSF.

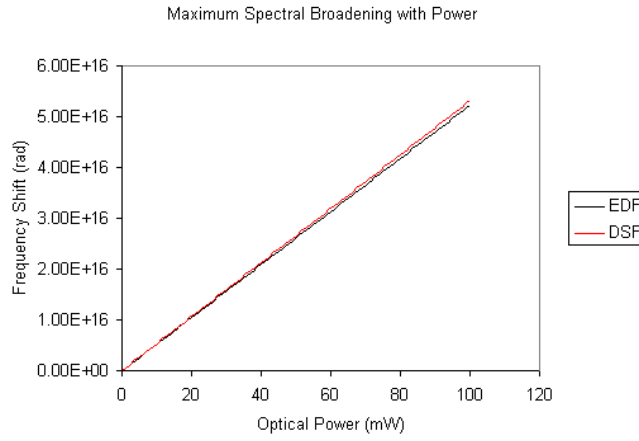


figure 3.7 Computational estimation of SPM induced spectral broadening over 116 m span of fibre.

From such comparisons *figure 3.7* we find that the magnitude of the maximum spectral shift in DSF and EDF over the same distance is similar for comparable powers. Therefore from a simple simulation we predict that an L-band amplifier can contribute significantly to SPM induced chirp.

To determine the L-Band amplifier contribution to system-wide FWM it is useful to compare the magnitude of the measured FWM products in a length of fibre to the FWM products at the amplifier output.

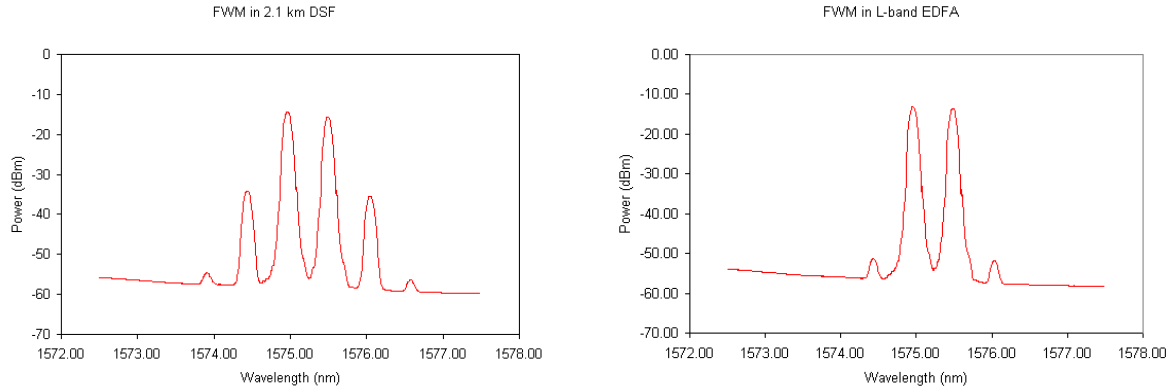


figure 3.8 Comparison of FWM in 2.1 km DSF and L-Band Amplifier, with 35 mW L-Band amplifier output.

It can be seen from these plots that the amplifier contribution to FWM is not small and therefore cannot be ignored when considering contributions to system performance.

3.2. Measurement of Scattering Effects

Measurement of scattering effects in a production EDFA is difficult since measurement techniques rely on detecting backscatter. Pre-built amplifiers will contain an isolator that prevents light exiting via the amplifier input. However, it may still be possible to extract some information about Brillouin scattering by comparing the amplifier output of a modulated signal with one that is not modulated. If SBS occurs within the amplifier then we would expect to see the output increase when the signal is modulated. Modulating the current across the source laser diode induces small changes in the laser cavity size, resulting in a dithered wavelength. The broadened signal is an inefficient pump for SBS and therefore reduces the effect. In a typical optical fibre, the Brillouin gain width is ~ 10 MHz, therefore modulation should broaden the signal >10 MHz for it to reduce its contribution to SBS.

The experimental arrangement shown in figure 3.9 has been used in order to detect SBS.

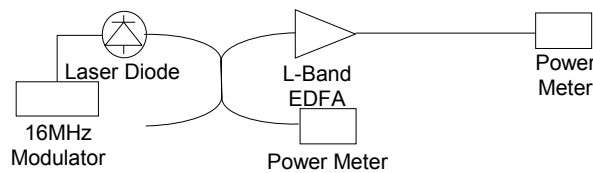


figure 3.9 Experimental arrangement to detect SBS in an optical fibre L-Band amplifier.

The results from our L-Band amplifier under test show no detectable SBS. This is not surprising since the amplifier fibre length may not be long enough to exhibit SBS. In standard fibres SBS occurs in kilometres of fibre length at typical optical powers of around 20mW.

From the simple relation for the Brillouin gain threshold g_B ,

$$P_{cr} \approx 21 \frac{A_{eff}}{g_B L_{eff}}$$

we can estimate the critical power at which SBS becomes a limiting factor in our *L*-band fibre amplifier. Assuming similar values to those of a typical fibre at 1550 nm, $A_{eff} = 50 \times 10^{-12} \text{ m}^2$, $g_B = 5 \times 10^{-11} \text{ m W}^{-1}$ and the amplifier effective fibre length $L_{eff} = 226 \text{ m}$ the critical power is expected to be, $P_{cr} \approx 93 \text{ mW}$. This is far higher than the maximum output of the amplifier tested here and therefore suggests that the limitations due to SBS are not important in *L*-Band amplifiers.

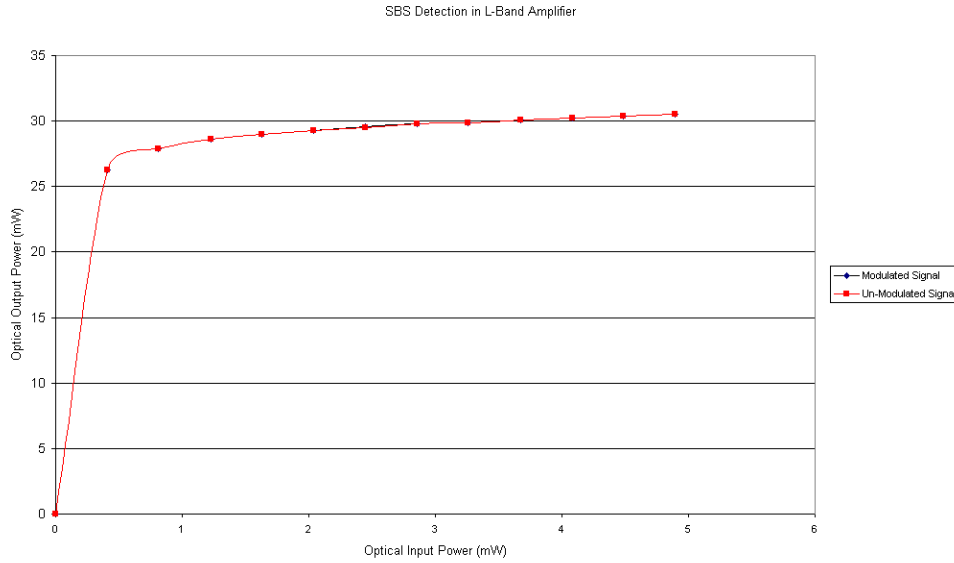


figure 3.10 Experimental results for the detection of SBS in an L-Band amplifier. The chart gives no indication that SBS was occurring within the amplifier.

In transmission fibre, SBS can inhibit Stimulated Raman Scattering (SRS) because the Raman threshold, P_{SRS} , is generally much higher than the SBS threshold. Therefore, SRS is not expected to be a problem within the *L*-Band EDFA. If we take a Raman gain coefficient $g_{SRS} = 7 \times 10^{-14} \text{ m W}^{-1}$, we can calculate the power threshold at which signal power would transfer exponentially to a Raman Stokes wave.

$$P_{SRS} \approx 32 \frac{A_{eff}}{g_{SRS} L_{eff}} \quad 3.4$$

Using the same input parameters as used to calculate the critical power for SBS we calculate an SRS threshold power of 197 W. An *L*-band amplifier is unlikely to be used at this sort of power level, and therefore we can ignore any contributions from this effect.

4. Conclusions

In our investigation we have considered the effects of fibre non-linearities in an L-Band Erbium Doped Fibre Amplifier. We have used computational modelling techniques to make predictions of non-linearity in L-Band amplifiers and made measurements on an example device.

It has been found that non-linear Kerr effects can occur significantly within the L-band amplifier, and in the case of SPM to the same magnitude as in DSF. Our L-band amplifier yields a comparable value of non-linear coefficient, and of non-linear phase shift when compared to dispersion shifted fibre.

From our measurements of FWM in the L-band amplifier we find the magnitude of the mixing products to be large when compared to predicted mixing products in an equivalent length of standard single mode fibre. Measurements of FWM in 2.1 km of transmission fibre show that FWM due to an L-Band amplifier is a significant contribution. Therefore, it is concluded that that FWM in L-Band EDFAs will contribute to cross-talk in multi-channel communication systems. This effect will be enhanced where channel spacing is small, and the FWM efficiency is high. Therefore in a further study Dense Wavelength Division Multiplexing (DWDM) systems should be considered. Longer lengths of EDF and higher powers would also enhance the detrimental effects of XPM and FWM. It is therefore recommended that further work be carried out characterising the non-linearities in long amplifiers with full WDM loading. For the purpose of further investigation an L-band amplifier should be constructed allowing complete characterisation of every aspect of the amplifier, and it's constituent parts. Of particular interest is the measurement of the effective area of pumped EDF at pump and signal wavelengths. This would enable accurate determination of the second order refractive index.

We now draw attention to the accuracy of the cw measurement of non-linear coefficient in pre-built amplifier units. We have no data to make comparison with, however we have assumed that equation 2.14 holds for a signal evolving along amplifying EDF. One possible way to test this may be to use the Frequency Resolved Optical Gating (FROG) method to obtain phase information directly from an optical pulse propagated through an active optical amplifier. Comparison with the original pulse would then lead to determination of the non-linear phase shift induced by the amplifying medium. Another method that deserves closer investigation is that of Near-field Scanning Optical Microscopy (NSOM) used to plot the electric field within a fibre at sub-wavelength resolution [18]. In a simpler approach we may be able to use an optical oscilloscope to observe the beat frequency between two closely spaced co-polarised signal wavelengths. By viewing the beat signal E_1 before it enters the amplifier, and then again after E_2 , we may obtain the amount of spectral broadening $\delta\omega$ and hence the non-linear phase shift, since $\delta\omega = \frac{d\phi_{NL}}{dt}$.

For example, assuming a sinusoidal beat frequency and ignoring dispersion:

$$E_1 = A \exp(i\omega t)$$

$$E_2 = A \exp(i\omega t) \exp(i\delta\omega t)$$

$$\therefore \delta\omega = \frac{-i}{t} \ln \frac{E_2}{E_1}$$

$$\Rightarrow \varphi_{NL} = \int_{t_{end}}^{t_{start}} \delta\omega dt$$

where $t_{end} - t_{start}$ is the envelope width.

The methods mentioned above would provide data to validate our approach to the measurement of the non-linear coefficient. It should be noted that in the example above the contribution from the mixing products to the SPM induced chirp is ignored.

In building a simulation we found there to be a requirement for many measured input parameters such as the fraction of the pump and signal wavelengths coupled into the doped region of EDF, the doping profile and the Er^{3+} emission and absorption cross-sections. The absorption cross-section is obtained through a relatively straight forward measurement, whilst it is more difficult to measure the emission cross-section directly, with it normally being calculated from the absorption by means of the McCumber relationship. Since these are parameters used by manufacturers in the design of amplifiers perhaps there is some requirement for their accurate and traceable measurement.

5. Acknowledgement

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Appendix A. Theory

i. Anharmonic Oscillator Model of Non-Linear Media

This model is commonly used to understand the origin of non-linear effects in optical fibres. The assumes a medium composed of N classical anharmonic oscillators per unit volume. Each oscillator describes physically an electron bound to an atom. The motion of an anharmonic oscillator in the presence of some driving force F with resonant frequency ω_0 is given by [19,20]:

$$\frac{d^2x}{dt^2} + \Gamma \frac{dx}{dt} + \omega_0^2 x + a_2 x^2 + a_3 x^3 = \frac{F}{m} \quad \text{A.i.i}$$

The first term is simply the acceleration from Newton's equation $F = ma$, the second term is a damping term, the third term describes the natural frequency of the oscillator and the fourth and fifth terms describe second and third order non-linearities. Since in fibre we are only really interested in 3rd order non-linearity we start by considering the equation

$$\frac{d^2x}{dt^2} + \Gamma \frac{dx}{dt} + \omega_0^2 x + a_3 x^3 = \frac{F}{m}. \quad \text{A.i.ii}$$

If the non-linearity is small, i.e. $\omega_0^2 x \gg a_3 x^3$, then we can solve A.i.ii by treating the non-linearity as a small perturbation to the linear equation for a forced harmonic oscillator

$$\frac{d^2x}{dt^2} + \Gamma \frac{dx}{dt} + \omega_0^2 x = \frac{F}{m}. \quad \text{A.i.iii}$$

Given the relation

$$F = q\mathbf{E} \quad \text{A.i.iv}$$

from Coulombs law, we assume an applied field $\mathbf{E}$ oscillating with driving frequency ω ,

$$\mathbf{E} = E e^{i\omega t} \quad \text{A.i.v}$$

and therefore induced motion of the electronic dipole

$$x(t) = x e^{i\omega t} \quad \text{A.i.vi}$$

that when substituted into eq. A.i.ii yields

$$\frac{d^2x}{dt^2} + \Gamma \frac{dx}{dt} + \omega_0^2 x + a_3 x^3 = -y\omega^2 e^{i\omega t} + i\Gamma y\omega e^{i\omega t} + \omega_0^2 y e^{i\omega t} + a_3 y^3 e^{i3\omega t}. \quad \text{A.i.vii}$$

It is apparent that in the presence of a strong applied field we will observe a 3rd harmonic. By using the same technique and with further iterations and two driving frequencies it can be shown that the generation of both sum and difference frequencies along with higher order harmonics is possible, thus showing the physical origin of effects such as harmonic generation and wave mixing.

The model leads directly to a description of the local dielectric polarisation $\mathbf{P}$.

ii. Non-Linear Propagation Equation

The propagation of light in optical fibres is governed by the wave equation as derived from Maxwell's equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{A.ii.i}$$

$$\nabla \times \frac{\mathbf{B}}{\mu_r \mu_0} = \epsilon_r \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{P}}{\partial t} \quad \text{A.ii.ii}$$

$$\nabla \cdot \mathbf{D} = 0 \quad \text{A.ii.iii}$$

$$\Rightarrow \nabla \cdot \mathbf{E} = 0$$

Taking the curl of A.ii.i

$$\begin{aligned} \nabla \times \nabla \times \mathbf{E} &= -\nabla \times \frac{\partial \mathbf{B}}{\partial t} \\ &\equiv \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla \times \frac{\partial \mathbf{B}}{\partial t} \end{aligned} \quad \text{A.ii.iv}$$

Then from A.ii.iii

$$\nabla^2 \mathbf{E} = \nabla \times \frac{\partial \mathbf{B}}{\partial t} \quad \text{A.ii.v}$$

and using A.ii.ii

$$\begin{aligned} \nabla^2 \mathbf{E} &= \mu_r \mu_0 \left[\epsilon_r \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{\partial^2 \mathbf{P}}{\partial t^2} \right] \\ &\equiv \nabla^2 \mathbf{E} = \mu_r \left[\epsilon_r \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2} \right]. \end{aligned} \quad \text{A.ii.vi}$$

By separating the polarisation component $\mathbf{P}$ into it's linear and non-linear parts

$$\mathbf{P} = \mathbf{P}_L + \mathbf{P}_{NL} \quad \text{A.ii.vii}$$

equation A.ii.vi expands to:

$$\nabla^2 \mathbf{E} = \mu_r \left[\epsilon_r \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu_0 \frac{\partial^2 \mathbf{P}_L}{\partial t^2} + \mu_0 \frac{\partial^2 \mathbf{P}_{NL}}{\partial t^2} \right] \quad \text{A.ii.viii}$$

In order to solve A.ii.viii we make some assumptions;

- $\mathbf{P}_{NL}$ is a small perturbation to $\mathbf{P}_L$
- Optical field polarisation is maintained along the x -axis to allow a scalar approach
- Quasi-monochromatic optical field

Here the slowly varying envelope approximation is used. This is valid in the case of a pulsed signal whose width ≥ 0.1 ps, or for two co-propagating beams of the same polarisation having a frequency separation such that the beat frequency $\Delta\omega \leq 10^{13} \text{ s}^{-1}$.

First we write the rapidly varying part of the electric field polarised along the x -axis as

$$\mathbf{E}(\mathbf{r}, t) = \hat{x} \frac{1}{2} [\bar{E}(\mathbf{r}, t) e^{-i\omega_0 t} + c.c.] \quad \text{A.ii.ix}$$

for the slowly varying envelope function $\bar{E}(\mathbf{r}, t)$ and complex conjugate $c.c.$. The polarisation components are treated similarly leading to:

$$P_L(\mathbf{r}, t) = \hat{x} \frac{1}{2} [\bar{P}_L(\mathbf{r}, t) e^{-i\omega_0 t} + c.c.] \quad \text{A.ii.x}$$

$$P_{NL}(\mathbf{r}, t) = \hat{x} \frac{1}{2} [\bar{P}_{NL}(\mathbf{r}, t) e^{-i\omega_0 t} + c.c.] \quad \text{A.ii.xi}$$

We transpose A.ii.ix into the frequency domain by the Fourier transform $\tilde{E}(\mathbf{r}, \omega - \omega_0)$,

$$\tilde{E}(\mathbf{r}, \omega - \omega_0) = \int_{-\infty}^{\infty} \bar{E}(\mathbf{r}, t) e^{i(\omega - \omega_0)t} dt \quad \text{A.ii.xii}$$

for which we are given a solution.

$$\nabla^2 \tilde{E} + \varepsilon(\omega) k_0^2 \tilde{E} = 0 \quad \text{A.ii.xiii}$$

where $k_0 = \omega/c$.

By separating variables and assuming a solution of the form

$$\tilde{E}(\mathbf{r}, \omega - \omega_0) = F(x) \tilde{A}(z, \omega - \omega_0) e^{i\beta_0 z} \quad \text{A.ii.xiv}$$

where $F(x)$ is the field distribution as a function of x across the fibre assuming circular symmetry. $\tilde{A}(z, \omega - \omega_0)$ is the slowly varying amplitude with z along the fibre and β_0 is the wave-number.

Ignoring $F(x)$ and substituting A.ii.xiv into A.ii.xiii, using the product rule we obtain an equation for $\tilde{A}$.

$$\begin{aligned}
& \nabla^2 \tilde{A} e^{i\beta_0 z} + \varepsilon(\omega) k_0^2 \tilde{A} e^{i\beta_0 z} = 0 \\
& \therefore \\
& \Rightarrow \nabla \tilde{A} e^{i\beta_0 z} = e^{i\beta_0 z} \frac{\partial}{\partial z} \tilde{A} + i \tilde{A} \beta_0 e^{i\beta_0 z} \\
& \Rightarrow \nabla^2 \tilde{A} e^{i\beta_0 z} = e^{i\beta_0 z} \frac{\partial^2}{\partial z^2} \tilde{A} + 2i \beta_0 e^{i\beta_0 z} \frac{\partial}{\partial z} \tilde{A} - \tilde{A} \beta_0^2 e^{i\beta_0 z} \quad \text{A.ii.xv} \\
& \therefore \\
& e^{i\beta_0 z} \frac{\partial^2}{\partial z^2} \tilde{A} + 2i \beta_0 e^{i\beta_0 z} \frac{\partial}{\partial z} \tilde{A} - \tilde{A} \beta_0^2 e^{i\beta_0 z} + \varepsilon(\omega) k_0^2 \tilde{A} e^{i\beta_0 z} = 0
\end{aligned}$$

Cancelling the exponential terms, and neglecting the term in $\frac{\partial^2}{\partial z^2} \tilde{A}$ since $\tilde{A}$ is a slowly varying function of z , yields:

$$2i \beta_0 \frac{\partial}{\partial z} \tilde{A} + [\varepsilon(\omega) k_0^2 - \beta_0^2] \tilde{A} = 0 \quad \text{A.ii.xvi}$$

We then make the substitution $\bar{\beta}^2 = \varepsilon(\omega) k_0^2$, followed by the approximation $\bar{\beta}^2 - \beta_0^2 = 2\beta_0(\bar{\beta} - \beta_0)$ where

$$\bar{\beta}(\omega) = \beta(\omega) + \Delta\beta \quad \text{A.ii.xvii}$$

hence we re-express A.ii.xvi as

$$\frac{\partial}{\partial z} \tilde{A} = i[\beta(\omega) + \Delta\beta - \beta_0] \tilde{A}. \quad \text{A.ii.xviii}$$

By expanding $\beta(\omega)$ as a Taylor series, neglecting third order terms and above due to our quasi-monochromatic assumption, A.ii.xviii becomes

$$\begin{aligned}
& \beta(\omega) = \beta_0 + (\omega - \omega_0) \beta_1 + \frac{1}{2} (\omega - \omega_0)^2 \beta_2 \\
& \therefore \\
& \Rightarrow \frac{\partial}{\partial z} \tilde{A} = i \left[(\omega - \omega_0) \beta_1 + \frac{1}{2} (\omega - \omega_0)^2 \beta_2 + \Delta\beta \right] \tilde{A} \quad \text{A.ii.xix}
\end{aligned}$$

Now, from the exponential term of the inverse Fourier transform

$$\begin{aligned}
& A(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{A}(z, \omega - \omega_0) e^{-i(\omega - \omega_0)t} d\omega \\
& \Rightarrow \Psi = e^{-i(\omega - \omega_0)t} \\
& \Rightarrow \frac{\partial}{\partial t} \Psi = -i(\omega - \omega_0) \Psi \quad \text{A.ii.xx} \\
& \therefore \\
& i \frac{\partial}{\partial t} = \omega - \omega_0,
\end{aligned}$$

we find that we may transform to the time domain t by way of the substitution

$$i \frac{\partial}{\partial t} = \omega - \omega_0 . \quad \text{A.ii.xxi}$$

Therefore substituting A.ii.xxi into A.ii.xix yields the propagation equation:

$$\begin{aligned} \frac{\partial A}{\partial z} &= i \left[\left(i \frac{\partial}{\partial t} \right) \beta_1 + \frac{1}{2} \left(i \frac{\partial}{\partial t} \right)^2 \beta_2 + \Delta\beta \right] \tilde{A} \\ \therefore \quad \frac{\partial A}{\partial z} &= -\beta_1 \frac{\partial A}{\partial t} - \frac{i}{2} \beta_2 \frac{\partial^2 A}{\partial t^2} + i\Delta\beta A. \end{aligned} \quad \text{A.ii.xxii}$$

The term in $\Delta\beta$ is responsible for non-linear effects, and when evaluated from the field distribution and non-linear refractive index², it is found to be

$$\Delta\beta = -i \frac{\alpha}{2} - \gamma |A|^2 . \quad \text{A.ii.xxiii}$$

Finally, by setting term $\beta_1 = 0$ therefore neglecting chromatic dispersion, and multiplying by i , the propagation equation becomes

$$\boxed{i \frac{\partial A}{\partial z} = \frac{1}{2} \beta_2 \frac{\partial^2 A}{\partial t^2} - \frac{i}{2} \alpha A - \gamma |A|^2 A.} \quad \text{A.ii.xxiv}$$

which, with $\alpha = 0$, is the non-linear Schrödinger Equation.

² Agrawal [2], pp. 34 - 43

iii. Beat Frequency Wave Mixing (FWM/ SPM)

Wave mixing can occur when two or more discrete wavelengths propagate within a fibre. Assuming that each signal is identically polarised then they will beat together forming an envelope oscillating at a ‘beat frequency’.

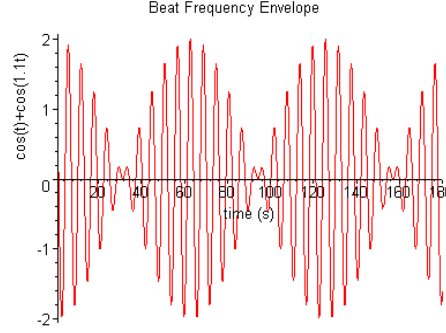


figure A.iv.i. Frequencies of 1 Hz and 1.1 Hz interfere by superposition to form an envelope carrier with frequency 0.1 Hz.

In fibres this is important since the optical field inside the fibre modifies the local dielectric polarisation field $\mathbf{P}$ at the frequency of the applied field, the electric dipoles then re-radiate at the same frequency. If two signals propagate with differing wavelengths/frequencies then the local dielectric polarisation will oscillate at the signal frequencies, and their beat frequency, re-radiating at a mix of the signal frequencies. This can be described as degenerate Four-Wave Mixing (FWM).

Quantum mechanically the process is described as the annihilation of two photons at pump frequencies ω_1 and ω_2 , and the simultaneous creation of two new photons at frequencies ω_3 and ω_4 . Conservation of energy dictates that the photon frequencies must satisfy

$$\omega_3 + \omega_4 = \omega_1 + \omega_2 . \quad \text{A.iii.i}$$

In fibres the contribution to this non-linear polarisation is governed by [21]

$$\mathbf{P}_{NL} = \epsilon_0 \chi^{(3)} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \mathbf{E}_i \mathbf{E}_j \mathbf{E}_k \quad \text{A.iii.ii}$$

where $\chi^{(3)}$ is the 3rd order susceptibility, and $\mathbf{E}_i$ is the electric field of the i^{th} signal out of n discrete monochromatic plane waves launched into the fibre.

If we ignore dispersion and set $n = 2$ we can assume that the field of each signal launched into the fibre with frequency ω is given by

$$\mathbf{E}_{i,j,k} = E_{i,j,k} \cos(\omega_{i,j,k} t) . \quad \text{A.iii.iii}$$

Setting $E_{i,j,k} = 1$, and substituting for $\mathbf{E}_{i,j,k}$ in A.iii.ii yields

$$\mathbf{P}_{NL} = \epsilon_0 \chi^{(3)} \begin{bmatrix} \cos^3(\omega_1 t) + 3 \cos(\omega_2 t) \cos^2(\omega_1 t) \\ + 3 \cos^2(\omega_2 t) \cos(\omega_1 t) + \cos^3(\omega_2 t) \end{bmatrix} \quad \text{A.iii.iv}$$

expanding we show that

$$\begin{aligned} \cos^3(\omega_1 t) &= \frac{3}{4} \cos(\omega_1 t) + \frac{1}{4} \cos(3\omega_1 t) \\ 3 \cos(\omega_2 t) \cos^2(\omega_1 t) &= 2(\cos[(2\omega_1 - \omega_2)t] + \cos[(2\omega_1 + \omega_2)t]) \\ 3 \cos(\omega_1 t) \cos^2(\omega_2 t) &= 2(\cos[(2\omega_2 - \omega_1)t] + \cos[(2\omega_2 + \omega_1)t]) \\ \cos^3(\omega_2 t) &= \frac{3}{4} \cos(\omega_2 t) + \frac{1}{4} \cos(3\omega_2 t) \end{aligned} \quad \text{A.iii.v}$$

and hence we have the following wave mixing products:

$$\begin{aligned} &\omega_1, \omega_2, \\ &3\omega_1, 3\omega_2, \\ &2\omega_1 - \omega_2, 2\omega_1 + \omega_2, \\ &2\omega_2 - \omega_1, 2\omega_2 + \omega_1. \end{aligned} \quad \text{A.iii.vi}$$

However, in reality we cannot neglect fibre dispersion as has been the case here. It is fibre dispersion that governs the magnitude of the four-wave mixing effect in fibres through the phase matching condition, i.e.

$$\begin{aligned} \Delta\beta &= \beta_3 + \beta_4 - \beta_1 - \beta_2 \\ &= \frac{(n_3\omega_3 + n_4\omega_4 - n_1\omega_1 - n_2\omega_2)}{c} = 0 \end{aligned} \quad \text{A.iii.vii}$$

FWM is most efficient when the phase mismatch, $\Delta\beta$ is zero.

The case whereby two discrete frequencies are launched into the fibre can be understood as degenerate FWM, i.e.

$$\omega = \omega_2 = \omega_3 \quad \text{A.iii.viii}$$

In this case the phase-matching condition is relatively easy to satisfy in optical fibres, and results in sidebands generated at frequencies predicted by A.iii.vi.

iv. Effective Area of a Gaussian Field Distribution

The effective area of an optical fibre is defined as:

$$A_{eff} = \frac{2\pi \left(\int_0^{\infty} |E(r)|^2 r dr \right)^2}{\int_0^{\infty} |E(r)|^4 r dr} = \frac{2\pi \left(\int_0^{\infty} I(r) r dr \right)^2}{\int_0^{\infty} I^2(r) r dr} \quad \text{A.iv.i}$$

Take a Gaussian electric field distribution $E(r)$ where:

$$E(r) = \exp\left(-\frac{r^2}{w^2}\right) \quad \text{A.iv.ii}$$

where w is the positive width parameter defined by the distance from the centre of the distribution at which $E(r)$ equals $1/e$.

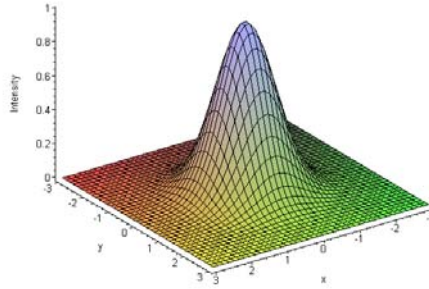


figure A.iv.i. Three dimensional Gaussian distribution with $w = 1$.

Assuming circular symmetry we may consider A.iv.ii in only one dimension. Making the substitution of A.iv.ii into A.iv.i yields the following integrals.

$$i_{top} = \int_0^{\infty} \left| \exp\left(\frac{-r^2}{w^2}\right) \right|^2 r dr \quad \text{A.iv.iii}$$

$$i_{bottom} = \int_0^{\infty} \left| \exp\left(\frac{-r^2}{w^2}\right) \right|^4 r dr \quad \text{A.iv.iv}$$

Starting with i_{top} we make the substitution

$$u = u(r) = \frac{-r^2}{w^2} \quad \text{A.iv.v}$$

re-arranging for r gives

$$r = r(u) = jw\sqrt{u} = jwu^{\frac{1}{2}} \quad \text{A.iv.vi}$$

where

$$j = \sqrt{-1} \quad \text{A.iv.vii}$$

we then differentiate A.iv.vi with respect to u yielding

$$r'(u) = \frac{jwu^{-\frac{1}{2}}}{2} \quad \text{A.iv.viii}$$

and perform the integration by substitution

$$\begin{aligned} \int f(r)dr &= \int f(r[u])r'(u)du \\ \Rightarrow \int \left| \exp\left(\frac{-r^2}{w^2}\right) \right|^2 r dr &= \int (jwu^{\frac{1}{2}}) \cdot |\exp(u)|^2 \cdot \left(\frac{jwu^{-\frac{1}{2}}}{2} \right) du \\ &= -\frac{w^2}{2} \int \exp(2u) du \\ &= -\frac{w^2}{4} \exp(2u) \end{aligned} \quad \text{A.iv.ix}$$

substituting for u in A.iv.ix

$$\int \left| \exp\left(\frac{-r^2}{w^2}\right) \right|^2 r dr = -\frac{w^2}{4} \left| \exp\left(-\frac{2r^2}{w^2}\right) \right| \quad \text{A.iv.x}$$

then introducing the limits of A.iv.iii

$$i_{top} = \int_0^\infty \left| \exp\left(\frac{-r^2}{w^2}\right) \right|^2 r dr = \frac{w^2}{4} \quad \text{A.iv.xi}$$

Performing this calculation for i_{bottom} using the same substitution yields the similar result

$$\int \left| \exp\left(\frac{-r^2}{w^2}\right) \right|^4 r dr = -\frac{w^2}{8} \left| \exp\left(-\frac{4r^2}{w^2}\right) \right| \quad \text{A.iv.xii}$$

$$i_{bottom} = \int_0^\infty \left| \exp\left(\frac{-r^2}{w^2}\right) \right|^4 r dr = \frac{w^2}{8} \quad \text{A.iv.xiii}$$

Therefore from A.iv.i, using A.iv.xi and A.iv.xiii we find the effective area of a fibre with a Gaussian field distribution to be:

$$A_{eff} = 2\pi \frac{i_{top}^2}{i_{bottom}} = 2\pi \frac{\left(\frac{w^2}{4}\right)^2}{\frac{w^2}{8}} \equiv \frac{16\pi w^4}{16w^2}$$

$$\therefore A_{eff} = \pi w^2$$

A.iv.xiv

Appendix B. Formulae

i. Attenuation (α)

$$\alpha(\text{neper / km}) = \frac{\ln(10)}{10} \alpha'(\text{dB / km}) \quad \text{B.i.i}$$

ii. Gaussian Curve

$$y = \exp\left(-\frac{x^2}{w^2}\right) \quad \text{B.i.ii}$$

w is a constant and is the width parameter.

iii. Angular Frequency (ω)

$$\omega = 2\pi\nu$$

$$\therefore$$

$$\nu = \frac{c}{\lambda}$$

B.i.iii

ν is frequency in Hz, λ is wavelength in nm and c is the speed of light in m s^{-1} .

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