

Calculating Uncertainties in Magnitude and Phase by Monte Carlo Simulation

**M Salter
NPL**

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Martin J Salter

Centre for Electromagnetic and Time Metrology, National Physical Laboratory, United Kingdom

1 Abstract

A method based on Monte Carlo Simulation (MCS) is described for calculating the uncertainties in the magnitude and phase of a complex-valued quantity from the uncertainties in its real and imaginary components. Some example 95% coverage intervals are presented and the coverage probability of the corresponding region in the complex plane is examined.

2 Cartesian and polar representations of a complex-valued quantity

Complex-valued quantities occur in the description of vibration and wave phenomena in areas of science and technology such as acoustics, optics and electromagnetics. For example in microwave metrology, the voltage reflection coefficients (VRCs) and voltage transmission coefficients (VTCs) of a 2-port device such as an attenuator are complex-valued quantities.

A complex-valued quantity Z can be represented in Cartesian form in terms of its real and imaginary components (X and Y respectively)

$$Z = X + jY$$

or in polar form in terms of its magnitude and phase (R and Φ respectively)

$$Z = R e^{j\Phi}$$

where $j = \sqrt{-1}$. If the complex number Z is regarded as a vector in the plane, the magnitude R is the length of the vector and the phase Φ is the angle the vector makes with the positive real axis measured in an anticlockwise sense. It follows that magnitude is non-negative and the phase of the origin is not defined. The Cartesian form is represented in the usual complex plane whereas the polar form is represented in the polar plane i.e. the plane with orthogonal magnitude and phase axes.

The two representations are related by the following transformation

$$\begin{aligned} X &= R \cos \Phi \\ Y &= R \sin \Phi. \end{aligned} \quad \dots\dots\dots(1)$$

This is a non-linear mapping from the half of the polar plane corresponding to positive magnitude ($R > 0$) to the punctured complex plane (the complex plane minus the origin). The transformation is periodic in Φ with a period of 360° . As a result, there are infinitely many points in the polar plane corresponding to each point in the complex plane and so the transformation is many-one. The transformation can be made one-one by restricting the mapping to the semi-infinite strip P in the polar plane defined by the inequalities

$$R > 0, \quad -180^\circ \leq \Phi < 180^\circ.$$

For each value of $R > 0$, the two points $[R, -180]^\dagger$ and $[R, 180]$ on the top and bottom edges of the strip P correspond to the same point in the complex plane. For this reason the top and bottom edges of P are identified point for point and P is a topological cylinder.

The restriction of the transformation in (1) to the semi-infinite strip P defines a one-to-one correspondence between P and the punctured complex plane. Its inverse can be written

[†] To distinguish between the two planes, a point $[R, \Phi]$ in the polar plane is written with square brackets and a point (X, Y) in the complex plane is written with round brackets.

$$\begin{aligned} R &= f_1(X, Y) = \sqrt{X^2 + Y^2} \\ \Phi &= f_2(X, Y) = \text{angle}(X, Y) \end{aligned} \quad (2)$$

where $\text{angle}(X, Y)$ is the angle in the range -180° to 180° between the vector $(X, Y)^T$ and the positive real axis measured in an anticlockwise sense. This inverse transformation assigns a unique magnitude and phase to each point in the complex plane apart from the origin. In the language of [1, 2], this is a multivariate model relating the two input quantities X and Y to the two output quantities (or measurands) R and Φ .

In many situations the polar form of a complex-valued quantity is more meaningful than the Cartesian form. For example, the phase of the VTC of a microwave device is related to the electrical length of the device and the magnitude is related to the loss of the device. No such direct physical interpretation can be given for the real and imaginary components. However, the expression of uncertainty in the polar representation is less straightforward than in the Cartesian representation. In principle, the real and imaginary axes extend indefinitely in both directions and so the marginal distributions of the real and imaginary components tend to be symmetric. On the other hand, magnitude is necessarily non-negative and so has a natural barrier at the origin. For this reason, the marginal distribution of magnitude tends to be skew particularly for points close to the origin. A further complication of the polar representation is that, since phase specifies the direction of a vector, it is defined on a circle and hence is cyclical. This suggests the strategy of first estimating the uncertainty in the Cartesian representation before transforming it to the polar representation. This paper describes a method of obtaining the uncertainty expressed in the polar representation from the uncertainty expressed in the Cartesian representation. The method is based on the propagation of the joint distribution of real and imaginary components to the joint distribution of magnitude and phase by means of Monte Carlo Simulation.

3 Numerical propagation of distributions by Monte Carlo Simulation

Suppose that the best estimate of the input quantities X and Y are $\bar{x}$ and $\bar{y}$ and the standard uncertainty in the input quantities is expressed by the covariance matrix V [3]. In accordance with the principle of maximum entropy [4], the input quantities X and Y are assigned a bivariate normal distribution with mean vector $\mu = (\bar{x}, \bar{y})^T$ and covariance matrix V . In general, V is a full matrix and the contours of the probability density function (pdf) are ellipses. For simplicity, in this paper attention will be confined to the case where the covariance matrix is a multiple of the identity matrix, $V = u^2 I$, where I is the 2×2 identity matrix. In this case, the elliptical contours degenerate into circles. A coverage region bounded by a contour of the pdf can be constructed corresponding to any given coverage probability. For a given coverage probability, the coverage region bounded by a contour of the pdf has the minimum area of all the coverage regions corresponding to that probability.

Monte Carlo Simulation (MCS) is a numerical method for approximating the distribution of the output quantities given a model and the distribution of the input quantities i.e. MCS is a numerical method of propagating distributions through a model. The application of MCS to the model (2) will now be described. A large random sample of size M ($M = 60,000$ say) is simulated from the bivariate normal distribution of X and Y [5]. This gives M points in the complex plane (x^i, y^i) for $i = 1$ to M . Applying the model (2) gives M points in the polar plane $[r^i, \phi^i]$ for $i = 1$ to M which constitute a random sample of size M from the joint distribution of the output quantities R and Φ where

$$\begin{aligned} r^i &= f_1(x^i, y^i) \\ \phi^i &= f_2(x^i, y^i) \end{aligned}$$

The magnitude values r^i and the phase values ϕ^i are now separately sorted into non-decreasing ordered lists $r^{(n)}$ and $\phi^{(n)}$. The ordered list $r^{(n)}$ provides an approximation to the cumulative distribution function (cdf) of the marginal distribution of R . Similarly, the ordered list $\phi^{(n)}$ provides an approximation to the marginal distribution of Φ . However, in the latter case the distribution is defined on a circle and so the concept of a cdf is not very helpful.

Best estimates of the output quantities ($\bar{r}$ and $\bar{\phi}$) will here be obtained by evaluating the model (2) at the best estimates of the input quantities ($\bar{x}, \bar{y}$)

$$\begin{aligned} \bar{r} &= f_1(\bar{x}, \bar{y}) \\ \bar{\phi} &= f_2(\bar{x}, \bar{y}) \end{aligned}$$

An alternative would be to use the means of r^i and ϕ^i for $i = 1$ to M as the best estimates of R and Φ respectively. Note that care needs to be taken when calculating the mean of a set of phase values [3]. In general these two methods of obtaining best estimates of the output quantities give different results [6].

A 95% coverage interval for magnitude can be obtained as the interval between the 2.5 percentile ($r^{(\text{int}(0.025M))}$) and the 97.5 percentile ($r^{(\text{int}(0.975M))}$) of the magnitude distribution (two-sided trimming) where $\text{int}(x)$ denotes the integer part of x . When $(\bar{x}, \bar{y})$ is close to the origin, the magnitude distribution is skew and it can happen that the above interval does not contain $\bar{r}$. In this case the coverage interval is taken to be the interval between zero and the 95 percentile ($r^{(\text{int}(0.95M))}$) (one-sided trimming). In general, the coverage interval constructed using the above method will not be the shortest possible interval with the required coverage probability.

A different method is needed to construct a coverage interval for phase because the distribution of phase is defined on a circle. One possibility is to proceed as follows. First locate the best estimate of phase ($\bar{\phi}$) in the ordered list of phase values. If it lies between the j th and the $(j+1)$ th members of the list $\phi^{(j)} \leq \bar{\phi} < \phi^{(j+1)}$, the lower and upper limits of the 95% coverage interval are given by $\phi_L = \phi^{(j_L)}$ and $\phi_U = \phi^{(j_U)}$ where $j_L = \text{int}(j - 0.475M)$ and $j_U = \text{int}(j + 0.475M)$. If either of the indices j_L or j_U are outside the range 1 to M then, since the distribution is defined on a circle, they can be brought back inside the range by addition or subtraction of M . If $j_L < 1$, $j_L := j_L + M$ and if $j_U > M$, $j_U := j_U - M$.

The coverage interval for magnitude defines an annular region in the complex plane with a coverage probability of 95% and similarly the coverage interval for phase defines a wedge shaped region in the complex plane lying between two rays with a coverage probability of 95% (Figure 1). The set of points common to both of these regions define a region in the complex plane having the shape of an "annular sector" (Figure 2) corresponding to a rectangular region in the polar plane. The annular sector is a subset of a 95% coverage region and so its coverage probability will be less than 95%. Since the annular sector is not bounded by a contour of the joint pdf of X and Y (and the corresponding rectangle in the polar plane is not bounded by a contour of the joint distribution of R and Φ), it will not have the minimum area property.



Figure 1 Regions in complex plane defined by 95% coverage intervals for magnitude and phase. These regions have coverage probabilities of 95%.

4 Results

Some example 95% coverage intervals for magnitude and phase calculated using the Monte Carlo scheme described above are shown in Table 1. All the points are on the positive real axis with a standard uncertainty in real and imaginary components (u) of 0.003. Also shown is the calculated coverage probability of the annular sector region. A graphical representation of magnitude and phase uncertainties for two points is shown in Figure 3. These results can be confirmed to be reasonable by numerical quadrature of the joint distribution of magnitude and phase which follows from the assumption of normally distributed real and imaginary parts with a covariance matrix equal to a multiple of the identity matrix. If intervals with a coverage probability of $\sqrt{0.95}$ are calculated

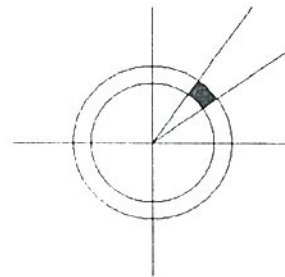


Figure 2 Annular sector region in the complex plane defined by coverage intervals for magnitude and phase.

for magnitude and phase, the resulting annular sector is found to have a coverage probability of approximately 0.95. This is in spite of the fact that magnitude and phase are not statistically independent of one another except in the case when the point coincides with the origin.

Real	Imag	Uncertainty (u)	Magnitude	Phase	End points of phase interval	Magnitude interval	Region probability
0.5	0	0.003	0.5	0	-0.7, 0.7	0.49, 0.51	0.903
0.3	0	0.003	0.3	0	-1.1, 1.1	0.29, 0.31	0.902
0.1	0	0.003	0.1	0	-3.3, 3.4	0.094, 0.106	0.903
0.003	0	0.003	0.003	0	-140.3, 140.4	0.0008, 0.0097	0.904
0.0015	0	0.003	0.0015	0	-161.7, 162.8	0.0007, 0.0086	0.903

Table 1 95% confidence intervals for magnitude and phase calculated using MCS. Also shown is the coverage probability of the annular sector region in the complex plane.

5 Conclusion

The skew and circular distributions which can occur for the magnitude and phase of a complex-valued quantity mean that the assignment of a normal or a Student's t distribution to the output quantities of model (2) can be inappropriate. It is for this reason that the use of propagation of distributions, implemented here using MCS, is particularly appropriate when calculating uncertainties in magnitude and phase.

6 Acknowledgements

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7 References

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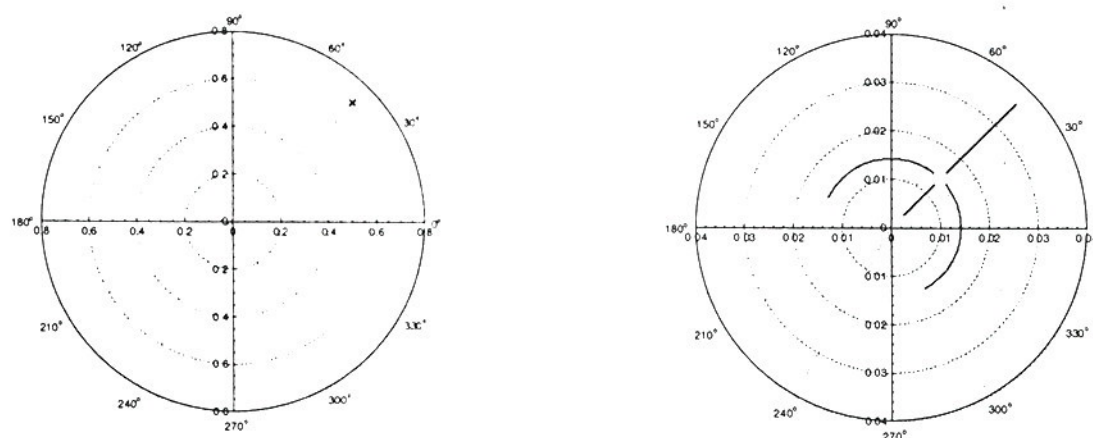


Figure 3 Graphical representation of the 95% coverage intervals for magnitude and phase obtained by MCS