



**ESTIMATION OF WEIBULL MODULUS FROM
DYNAMIC AND STATIC FATIGUE EXPERIMENTS**

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Abstract

In the statistical analysis of strength data on brittle materials it is useful to estimate the range of uncertainties in the calculation of parameters which characterise strength distributions, such that the reliability of subsequent use of the parameters can be judged. This has been performed for simple short-term strength distributions before, but in this report we develop a method for analysing static and dynamic fatigue data and evaluate it using a Monte Carlo simulation based on a single Weibull distribution of flaw sizes and hence inert strengths. It is shown that the technique is appropriate for analysing 50 dynamic fatigue results over four decades of stressing rate, and gives a Weibull modulus with a confidence level similar to that for a randomly chosen set of simple strength test specimens. Similar results were obtained with an analysis of static fatigue tests. The method was applied to real experimental data for soda-lime glass and provided a reasonable match between the Weibull modulus for conventional strength testing and those for static and dynamic fatigue.

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1. INTRODUCTION

Ceramic materials are susceptible to delayed failure or degradation of their mechanical strength under sustained loading as a consequence of subcritical crack growth. The rate subcritical crack growth is generally considered to have three regions of dependence on the stress intensity factor, K , at the crack-tip [1,2] corresponding with slow, environmentally controlled growth (Region I), environment access rate limited growth (Region II), and environment independent growth (Region III, near K_{IC}), although this is not always detectable in practice. An effective threshold for detectable crack growth exists and the lower end of Region I, often called the fatigue limit. The dominant behaviour in laboratory testing over periods in excess of a second or two, and that which essentially controls delayed failure, corresponds to what is known as "Region I", where the crack velocity, v , is often represented by a logarithmic dependence on the stress intensity factor, K [3]:

$$v = AK^n \quad (1)$$

where A is a constant and n is known as the subcritical crack growth exponent. In disregarding the other Regions, it is assumed that they contribute negligibly to the overall behaviour in terms of time.

The techniques that are used to study subcritical crack growth behaviour in ceramics basically fall into two categories: those based on the growth of long, induced cracks ($> 100 \mu\text{m}$), and those based on short, natural flaws ($< 100 \mu\text{m}$). Experiments involving long cracks enable the direct observation of the growing crack and the evaluation of the crack velocity, v , as a function of the applied K . In contrast, those using short, and usually unobservable cracks, involve either long-term tests to measure the time-to-failure in "static" fatigue tests [4-6] or determining the effect of subcritical behaviour on the strength of specimens stressed at different rates, in what are known as "dynamic" fatigue tests [7,8]. The results of tests performed using natural flaws are subject to the influence of the statistical distribution of the critical flaw size distribution between specimens, whereas in long crack experiments the statistical variation of the data is primarily a result of spatial variation in microstructure and measuring errors. There are, however, disadvantages in using long crack experiments; they are difficult to perform since they require the stable growth of long cracks in brittle materials, and the behaviour of large cracks may not imitate that of short, natural flaws, especially in coarse-grained [9-13] and transformation-toughened materials [14-16].

The scatter of the data in static and dynamic fatigue experiments, which is controlled by the statistical distribution of the critical flaw sizes of, and positions within, the test specimens, is typically large. Whilst this statistical scatter presents a problem in determining accurate parameters for subcritical crack growth, it obviously contains valuable information about the statistical strength variation of the material being tested. If we assume, as is commonly done, that the variation in strength is described by Weibull statistics [17], it must in principle be possible to extract the Weibull modulus, m , from the same data. In this paper, computational and experimental data are presented which indicate that usefully accurate values for the Weibull modulus can be obtained from dynamic and static fatigue experiments. The method used involves converting the fatigue data into a normalised strength distribution, from which m is obtained by a standard Weibull analysis. The error associated with the values obtained, as a function of the testing conditions used and the number of specimens tested, is quantified

using Monte-Carlo simulations. Although it is possible to analyse n in the same way as m we have not presented these results since a thorough analysis of this has already been performed by Ritter et al. 1981 [19], and our results are in good agreement with theirs.

2. THEORY

2.1 Computationally generated data

The probability of failure, P_f , of a material which obeys unimodal Weibull strength statistics has the following dependence on the applied stress [3,17,18]:

$$P_f \propto 1 - \exp - \left(\frac{\sigma - \sigma_t}{\sigma_o} \right)^m \quad (2)$$

where σ is the applied stress, σ_t represents the threshold stress below which failure does not occur, and σ_o is the normalising stress, sometimes called the characteristic strength. If identical specimen geometries and testing conditions are used then the failure stress, σ_f , corresponding a particular failure probability, P_f , can be related to the failure stress at another failure probability, say the median probability, $P_f = 0.5$, as follows:

$$\frac{\sigma_f - \sigma_t}{\sigma_{0.5} - \sigma_t} = \left(\frac{\log (1 - P_f)}{\log 0.5} \right)^{1/m} \quad (3)$$

It is usually assumed in an initial Weibull analysis that there is no strength threshold, i.e., $\sigma_t = 0$, or that its effect is insignificant, on the grounds that there is no reason to assume an upper limit to initial flaw size, so that:

$$\frac{\sigma_f}{\sigma_{0.5}} = \left(\frac{\log (1 - P_f)}{\log 0.5} \right)^{1/m} \quad (3)$$

If a lower strength limit does exist for a particular material, Eqn (4) is still valid so long as data from near the threshold are not used. Eqns (3) and (4) can be used to generate normalised strength distribution data for different m -values by simply randomly generating $P_f = 0$ to 1. These strength values are related to the toughness and failure flaw size, a_f , as follows:

$$K \propto \sigma_f a_f^{1/2} \quad (5)$$

So that:

$$\frac{a_f}{a_{0.5}} = \left(\frac{\log 0.5}{\log (1 - P_f)} \right)^{2/m} \quad (6)$$

The failure stress in dynamic fatigue, σ_{fd} is related to the stressing rate, β , as follows [7,8]:

$$\sigma_{fd} \propto a_i^{\frac{1-n/2}{n+1}} \beta^{\frac{1}{n+1}} \quad (7)$$

where a_i is the initial size of the critical flaw. It is therefore a simple matter to generate randomly a computational set of dynamic fatigue data based on Weibull statistics for a specific value of n ; the first term in Eqn (7) accounts for the statistical variation in flaw size and the second term the dependence on the stressing rate. Substituting Eqn (6) into Eqn (7) for a constant stressing rate gives:

$$\frac{\sigma_f}{\sigma_{0.5}} = \left(\frac{\log(1 - P_f)}{\log 0.5} \right)^{\frac{n-2}{m(n+1)}} \quad (8)$$

The fact that Eqns (8) and (4) are different warrants some comment. The Weibull strength distribution given by Eqn (4) is based on the failure of the specimens without any subcritical crack growth occurring, i.e. their strength is uniquely defined by the size of the critical flaw in the pretested specimen. Eqn (8), however, is based on an analysis of the effect of subcritical crack growth on failure strength and therefore accounts for the different failure scenarios. The two equations are in fact very similar when n is very large, >20 , which is typical for most ceramic materials. If this condition is not satisfied then strength distribution data would need to be analysed based on Eqn (8), and not Eqn (4) as it is commonly done, in order to obtain the Weibull modulus, m .

Static fatigue data can be computationally generated in the same way. The time-to-failure has the following dependence [3,5,6]:

$$t_f \propto \frac{K_i^{2-n}}{\sigma^2} \quad (9)$$

which can be rewritten as:

$$t_f \propto \frac{a_i^{\frac{2-n}{n}}}{\sigma^n} \quad (10)$$

or:

$$\sigma \propto t^{-1/n} \quad (11)$$

An equation equivalent to Eqn (8) can be obtained for constant time-to-failure in static fatigue experiments:

$$\frac{\sigma_f}{\sigma_{0.5}} = \left(\frac{\log(1 - P_f)}{\log 0.5} \right)^{\frac{n-2}{mn}} \quad (12)$$

As for Eqn (8), Eqn (12) differs from Eqn (4) because it accounts for the influence of subcritical crack growth on failure strength, but for $n > 20$ the difference is small.

The derivations used to analyse dynamic and static fatigue data are based on the following assumptions:

crack growth behaviour is described by Eqn (1);

$\sigma_i = 0$;

$\sigma < 0.9 \sigma_s$ [3,5-8], where σ_s is the inert strength (fast fracture with no subcritical crack growth).

2.2 Analysis of static and dynamic fatigue data to obtain n and m

The stress exponent, n , is determined from the slope of the curve obtained from a least squares regression fit to logarithmically plotted data for dynamic and static fatigue. The analysis for m is more complicated and requires a normalisation of all of the data points, using the calculated value of n , so that they are effectively equivalent to a strength distribution for a standard strength test. For dynamic fatigue data this requires the failure stress to be scaled according to some arbitrary, normalising, stressing rate using Eqn (7) and the value of n obtained experimentally by fitting a line to all the data points. The analysis of m for static fatigue data cannot be normalised in terms of a stressing rate, instead, we have to normalise against some arbitrary time-to-failure, say one second, using Eqn (11), again using a value of n calculated from the experimental data.

Having normalised the dynamic and static fatigue data in terms of equivalent strength distribution data it is a simple matter to obtain the Weibull modulus using a standard Weibull plot of $\log \log(1/1 - P_{fi})$ versus $\log(\sigma_{fi})$, where P_{fi} is the probability of failure of the i th ranked sample of strength σ_{fi} out of a total of N specimens given by:

$$P_{fi} = \frac{i}{N + 1}$$

This is, of course, only one of several estimators that are commonly used but, being simple, serves adequately for the purposes of demonstration. If other estimators were to be used, different levels of bias in m might arise.

Using a Monte-Carlo simulation it is possible to assess the accuracy and error associated with determining n and m from dynamic and static data. This is something which could not be done effectively using real experimental data because of the large number of tests that would be required and the uncertainty in the true values of n and m for a real material.

3. EXPERIMENTAL DETAILS

To supplement the computational analysis, the procedure for obtaining m from real experimental data from dynamic and static fatigue experiments was also investigated. Soda-lime glass bars, with dimensions of 3 x 4 x 50 mm, were tested in four-point bending jigs with inner and outer spans of 20 and 40 mm respectively. An experimental value of the m was initially obtained by performing a standard series of strength tests on a batch of 30 specimens under identical conditions in constant stress rate tests. Identical batches of specimens were then used to generate sets of dynamic and static fatigue data using the same flexure jig arrangement that was used for the strength distribution tests. These sets of data were then analysed in the same way as the computationally generated data, and n and m were obtained.

4. RESULTS AND DISCUSSION

4.1 Computationally generated data and Monte-Carlo simulation

Figures 1a and 2a show typical examples of computationally generated dynamic and static fatigue¹ data for $n = 20$, $m = 6$ and a total specimen batch size of $N = 30$ divided equally among five different loading rates or times to failure. By a least squares fit of this type of data an estimated value of n can be obtained. In these particular examples, values of n of 28 and 24 were obtained from the dynamic and static fatigue data respectively, the difference from the original value of 20 being a result of the random selection of starting "results". Figures 1b and 2b show the same data now converted using Eqns (7) and (11) to give normalised strength distributions. Figures 1c and 2c are the corresponding Weibull plots. In these example sets of data the estimated values of m happen to be very close to the value of 6 used to generate the original data sets; 6.1 and 6.0 for the dynamic and static data respectively.

By repeating the above procedure for a large number of batches (1,000) of randomly generated data it was possible to determine the possible error associated with calculating m from dynamic and static fatigue data as a function of the number of specimens in a test batch. The error involved is obviously dependent on n , m , N , the number of decades of stressing rate, β , in dynamic fatigue data, and on the number of decades of time-to-failure, t_p , in static fatigue data. In order to keep the number of permutations to a manageable level typical values for these variables were set: $n = 20$ and 100; $m = 5$ and 15; four decades of stressing rate in dynamic fatigue; five decades of time-to-failure in static fatigue; and N was variable from 4 to 50.

¹In this particular case it will be noted that the generated data are a series of stress levels that would produce given times of failure. In practice, of course, one would fix the stress levels and record times to failure. If this latter method is employed, to a first approximation it is still necessary to reduce all the data to a single time to failure for determining the Weibull distribution. It has been shown that equivalent results are produced.

For any one set of n , m , N and testing conditions, the variation of the calculated values of m , although slightly skewed, approximated a Gaussian distribution (Figure 3a). It was therefore appropriate to quantify the estimated errors in terms of simple coefficient of variation. The mean of the distributions, however, did not correspond to the value of m used to generate the random sets of data. It was biased to a value less than the true value, with the error decreasing with increasing sample size. The biased error for a sample batch of 10 specimens was about 10%, decreasing to about 3% for 200 specimens (Figure 3b). The bias and error associated with calculating m using different estimators has been reported elsewhere [20]. Using a standard curve, such as Figure 3b, it is a simple matter to remove this bias.

Figures 4 and 5 show the results of the analysis of the coefficients of variation for the estimation of m from dynamic and static fatigue data, respectively. The coefficient of variation appears to be insensitive to n and m . For a specimen batch of 50 it is ~ 0.1 to 0.15 , which is similar to values obtained from Monte-Carlo simulations of a single batch of strength data (Figure 4a).

4.2 Experimental data

Figures 6a and 7a show real experimental data for dynamic and static fatigue experiments on soda-lime glass. Figures 6b and 7b are the corresponding Weibull plots of the normalised data. The Weibull parameters obtained were 11.3 and 7.7 for the dynamic and static fatigue data respectively. Figure 8 is a Weibull plot for the results of the strength tests performed on an identical batch of specimens in order to obtain an experimental value of m , 8.0 for this particular material. All of the values of m overlap within the window of their coefficients of variation, which are ~ 0.2 for the number of specimens tested. The values of m quoted above should be increased by $\sim 10\%$, as demonstrated by Figure 3(b), to remove the bias introduced by the estimator in the Weibull analysis [20].

The difference in the values of m obtained are quite large, and would significantly influence the results of probability of failure calculations. If the data presented in Figure 4a are compared with those reported in [20] it becomes clear that the uncertainty in calculating m from dynamic or static fatigue data sets is very similar to that obtained from Monte Carlo estimates of Weibull moduli for fast fracture strength alone. That is, the potential error in n is not a significant factor in determining the uncertainty in m . Thus, while this demonstrates the potential of determining both n and m from static or dynamic fatigue data, and avoiding the separate determination of m , the difficulty of obtaining usefully reliable values of m for engineering design remains.

5. CONCLUSIONS

A computational method has been described which enables an estimate of the Weibull modulus to be made from dynamic and static fatigue data. The method involves a normalisation of the data to obtain a strength distribution. The errors associated with the estimated values of m have been quantified using a Monte-Carlo simulation and have been found to be very similar to those for simulations of fast fracture testing. The results obtained

on computationally generated data indicate that m can be estimated to a useful accuracy. For example, for a dynamic fatigue test performed using 50 specimens over 4 decades of stressing rate, the coefficient of variation of the estimated m would be ~0.10 to 0.15, which is the same as that to be expected from a set of 50 strength test results determined at a given stressing rate. It is therefore possible to employ sets of static or dynamic fatigue data for the calculation of Weibull modulus to the same level of reliability as would be achieved for a single batch of strength results for the same total number of test-pieces.

The method assumes that the data generated by dynamic and static fatigue are controlled by Region I subcritical crack growth behaviour. This assumption is realised for most ceramic materials in the range of stressing rates and times-to-failure practically achievable in dynamic and static fatigue tests. However, if this is not the cases, then it should be possible to observe a plateau in the experimental data, and results from this region could simply be ignored.

The method has been applied to real experimental data generated for a soda-lime glass, and the results obtained for m are in good agreement with the value obtained from a standard Weibull analysis of strength data.

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REFERENCES

1. S.M.Wiederhorn, J.Amer.Ceram.Soc., 1967, **59**, 407.
2. S.M.Wiederhorn, S.W.Freiman, E.R.Fuller, C.J.Simmons, J.Mater.Sci., 1982, **17**, 3460.
3. R.W.Davidge, J.R.McClaren and G.Tappin, J.Mater.Sci., 1973, **8**, 1699.
4. S.M.Wiederhorn and L.H.Bolz, J.Amer.Ceram.Soc., 1970, **53**, 543.
5. A.G.Evans, Int.J.Fract. , 1973, **9**, 267.
6. A.G.Evans and S.M.Wiederhorn, Int.J.Fract., 1974, **10**, 379.
7. R.J.Charles, J.Appl.Phys., 1958, **29**, 1657.
8. A.G.Evans, Int.J.Fract., 1974, **10**, 251.
9. H.Hübner and W.Jillek, J.Mater.Sci., 1977, **12**, 117.
10. P.L.Swanson, C.J.Fairbanks, B.R.Lawn, Y-W.Mai and B.J.Hockey, J.Amer.Ceram.Soc., 1987, **70**, 279.
11. Y-W.Mai and B.R.Lawn, J.Amer.Ceram.Soc., 1983, **70**, 289.
12. S.J.Bennison and B.R.Lawn, Acta.Metall., 1989, **37**, 2659.
13. A.Reichl and R.W.Steinbrech, J.Amer.Ceram.Soc., 1988, **71**, C299.
14. R.C.Garvie, R.H.Hannink and R.T.Pascoe, Nature, 1975, **258**, 703.
15. M.V.Swain in *Fracture Mechanics of Ceramics Vol 6*, edited by R.C.Bradt, A.G.Evans, D.P.H.Hasselman and F.F.Lange, Plenum Press, New York, 1986, p353.
16. R.A.McMeeking and A.G.Evans, J.Amer.Ceram.Soc., 1982, **65**, 242.
17. W.Weibull, Ing. Veternskaps Akad. APYEA, Handlinger, 1939, **155**, 5.
18. W.Weibull, J.Appl.Mech., 1951, **18**, 293.
19. J.E.Ritter, N.Bandyopadhyay and K.Jakus, Amer.Ceram.Soc.Bull., 1981, **60**, 798-806.
20. R.Langlois, J.Mater.Sci.Lett., 1991, **10**, 1049-1051.

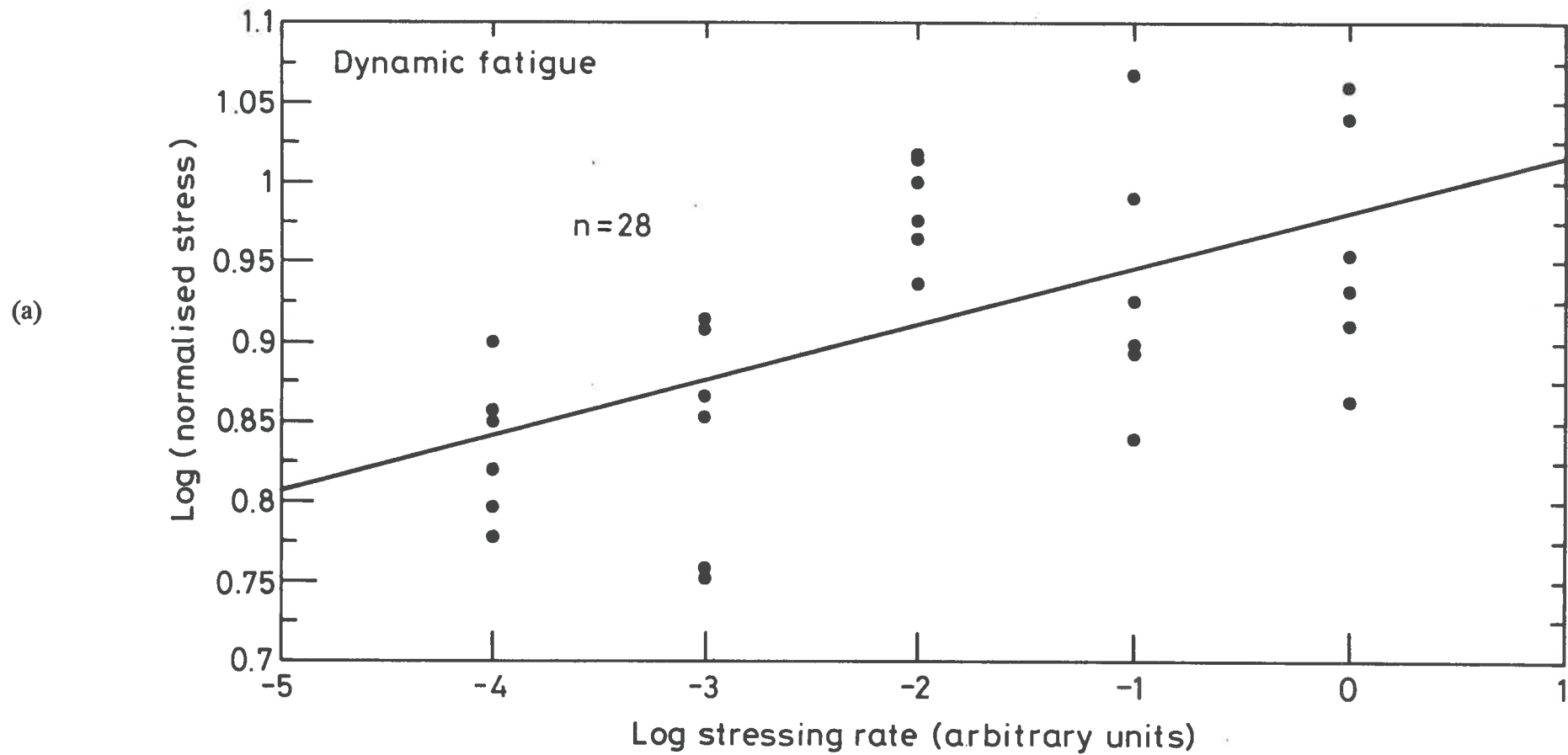


Figure 1. (a) Computationally generated dynamic fatigue data for $n = 20$, $m = 6$ and $N = 30$; (b) data normalised to give strength distribution; (c) Weibull plot.

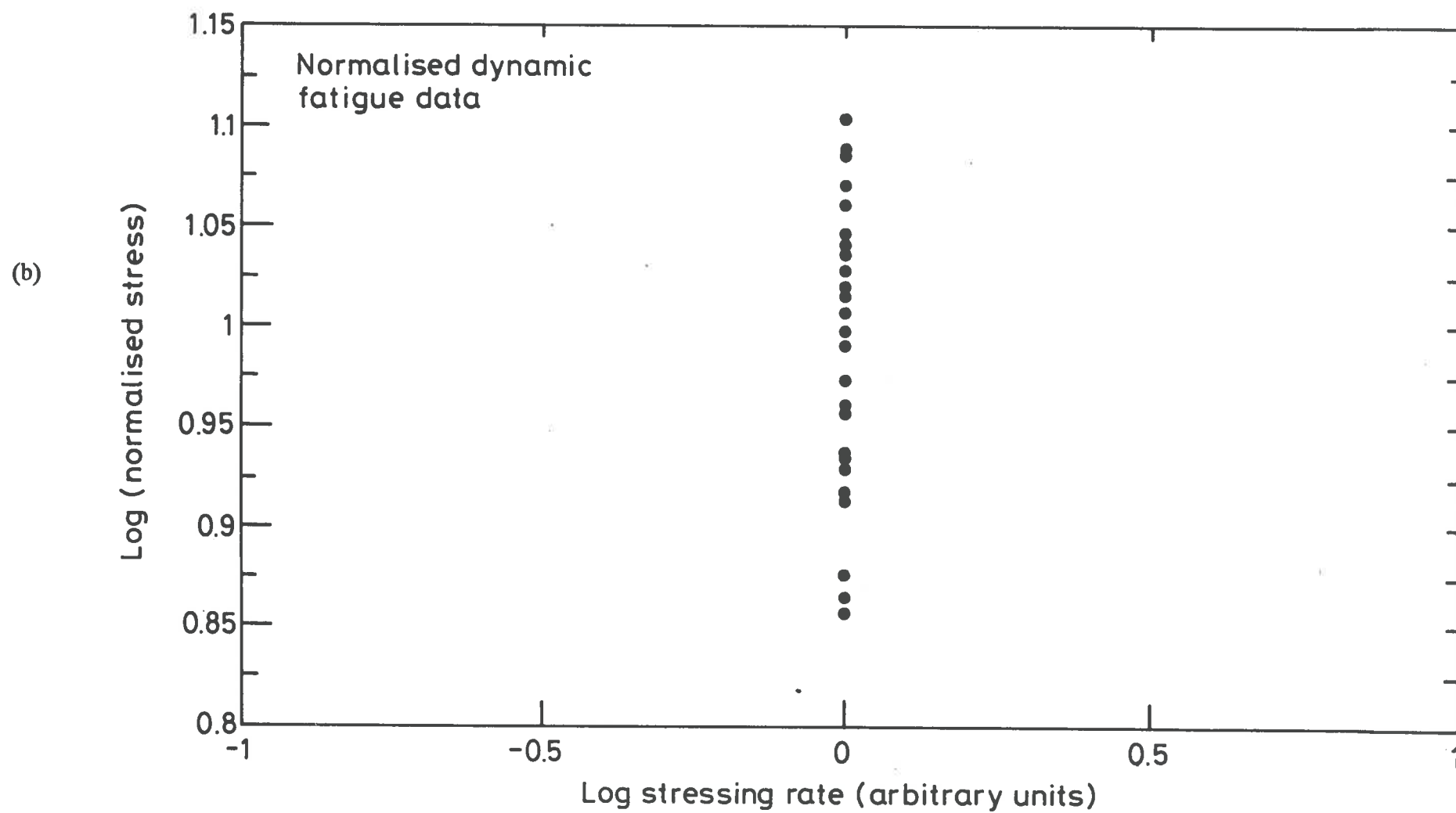


Figure 1(b)

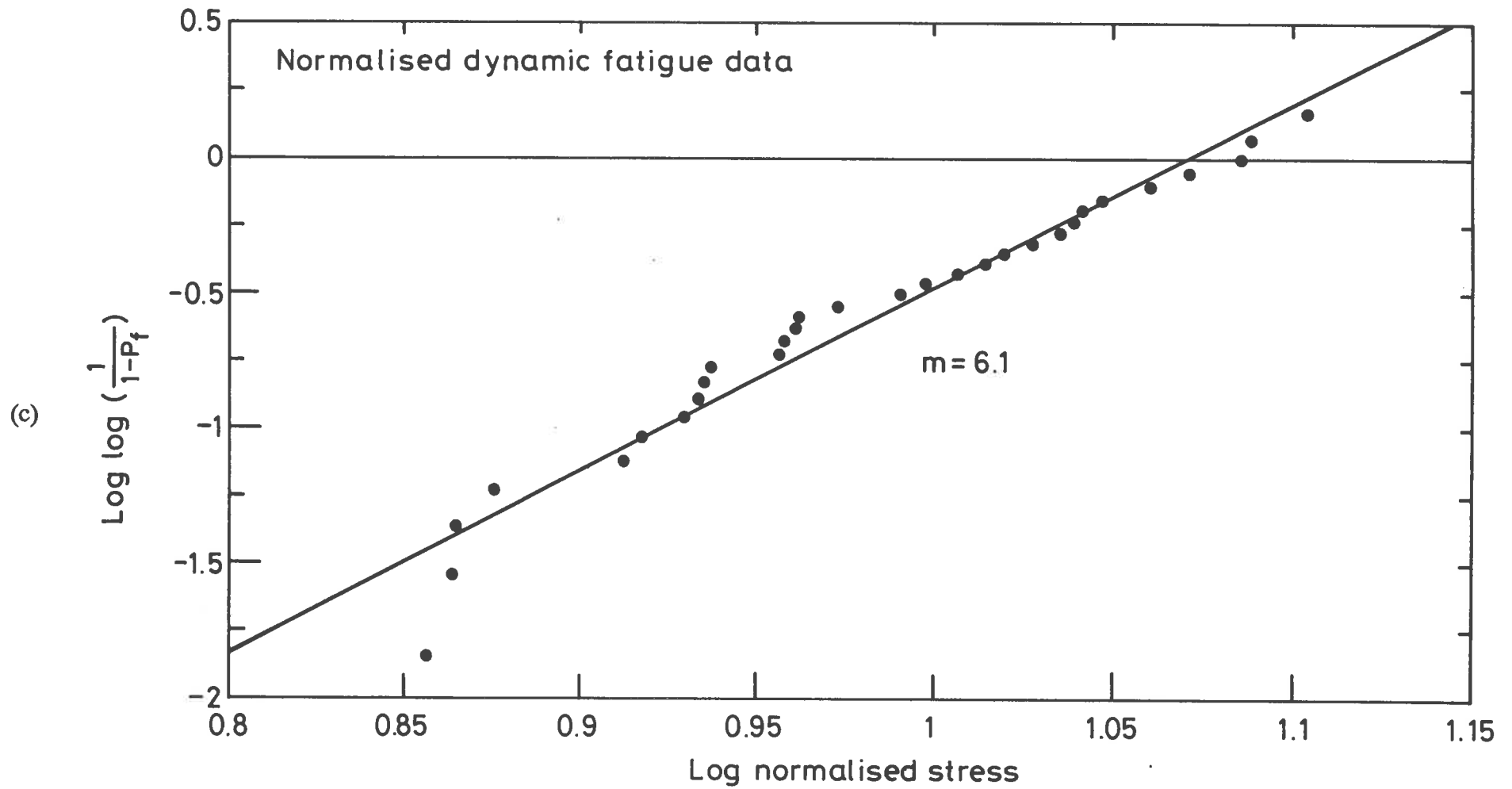


Figure 1(c).

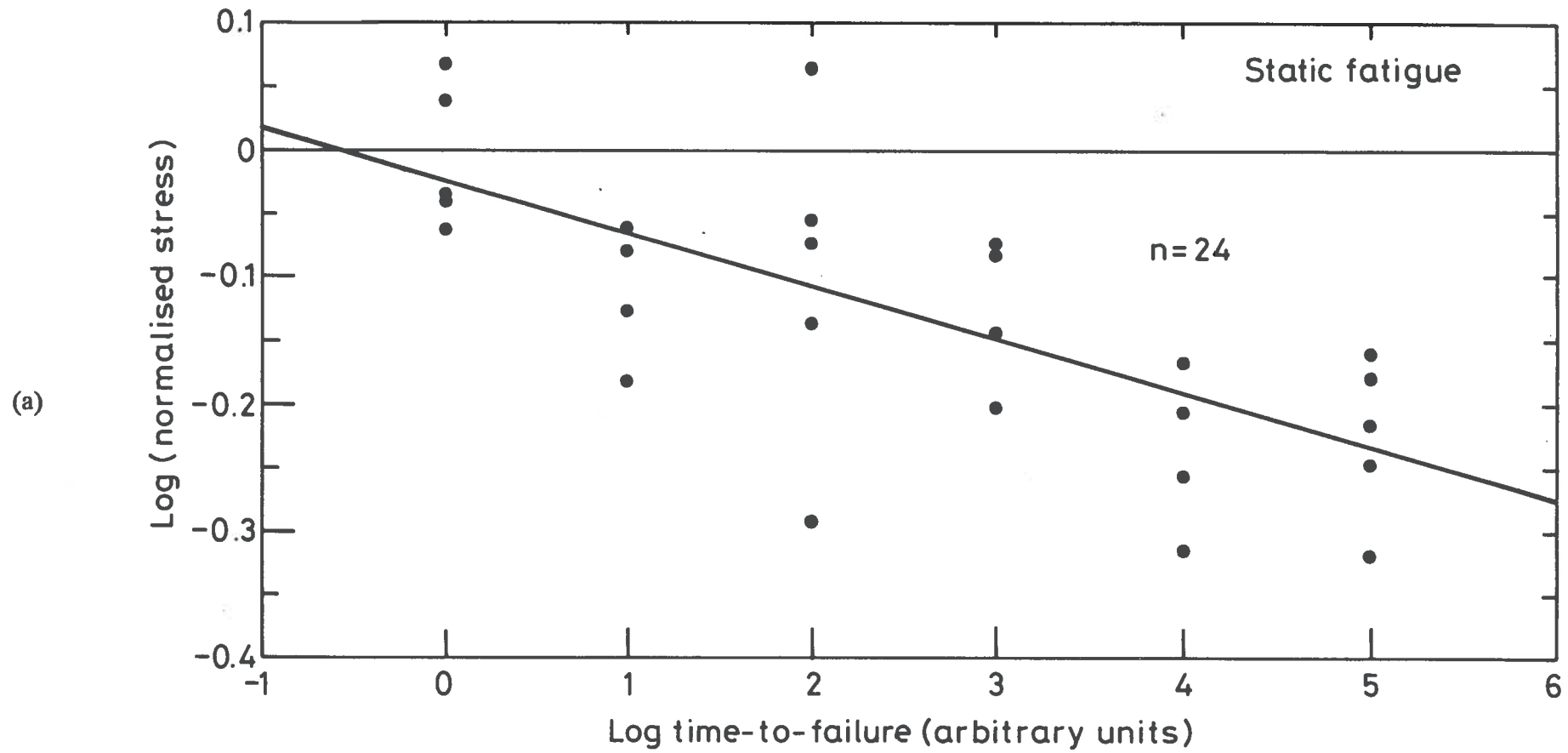


Figure 2. (a) Computationally generated static fatigue data for $n = 20$, $m = 6$ and $N = 30$; (b) data normalised to give strength distribution; (c) Weibull plot.

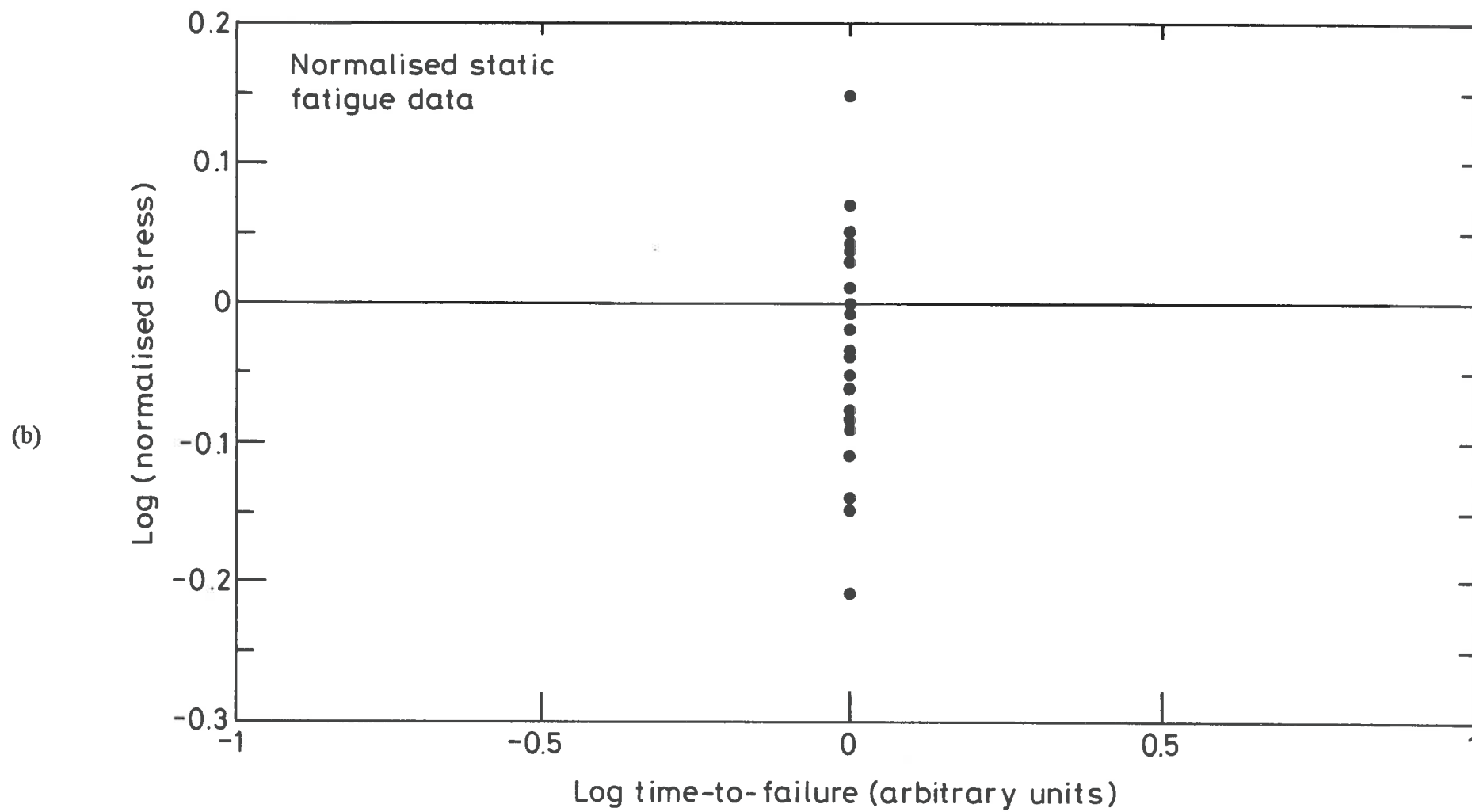


Figure 2(b)

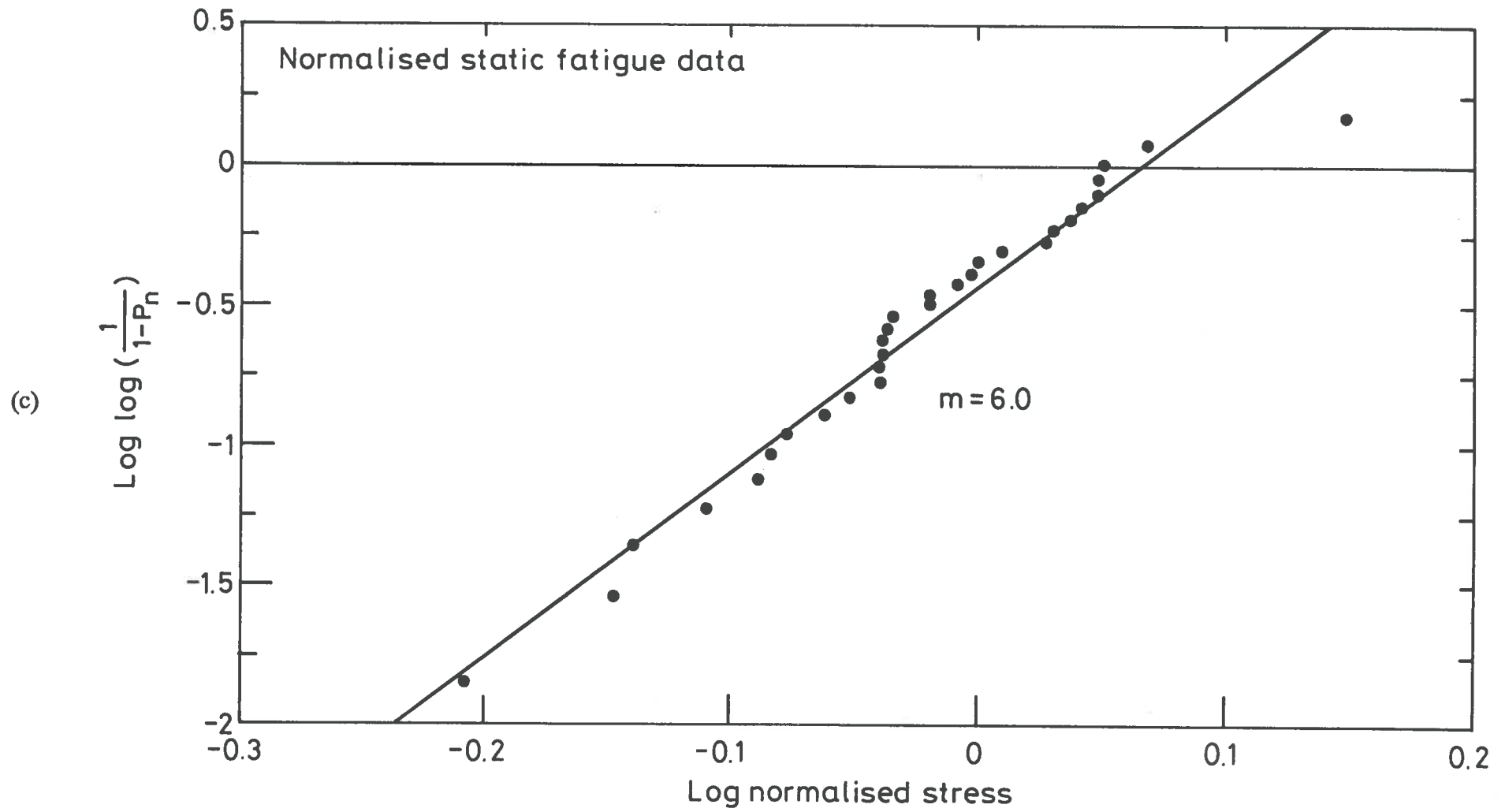


Figure 2(c)

(a)

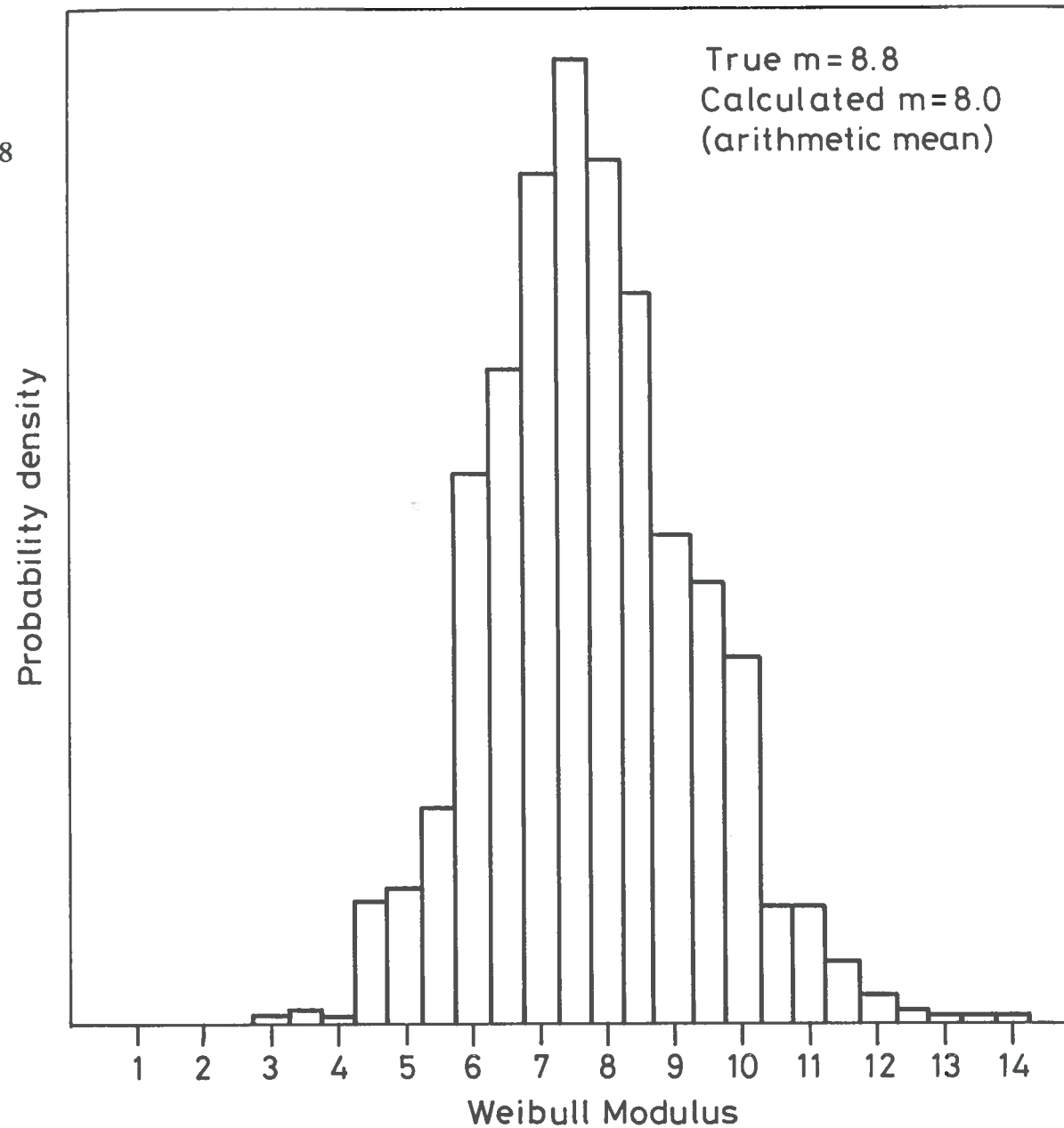


Figure 3. (a) Distribution of estimated values of m determined from analysis of computationally generated strength data for $m = 10$ (1,000 iterations). (b) Arithmetic mean of calculated Weibull modulus as a function of specimen batch size for a true $m = 5$.

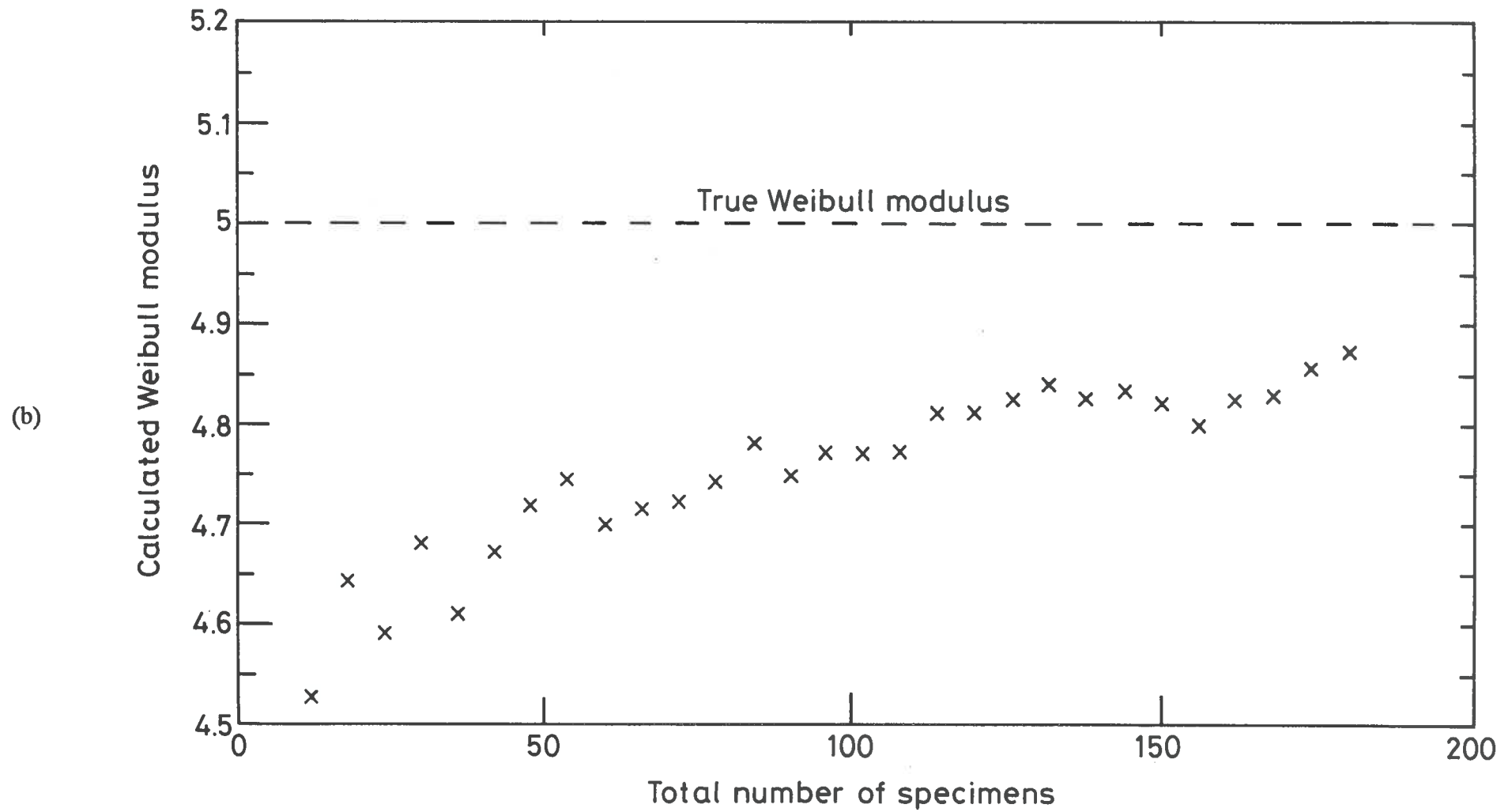


Figure 3(b)

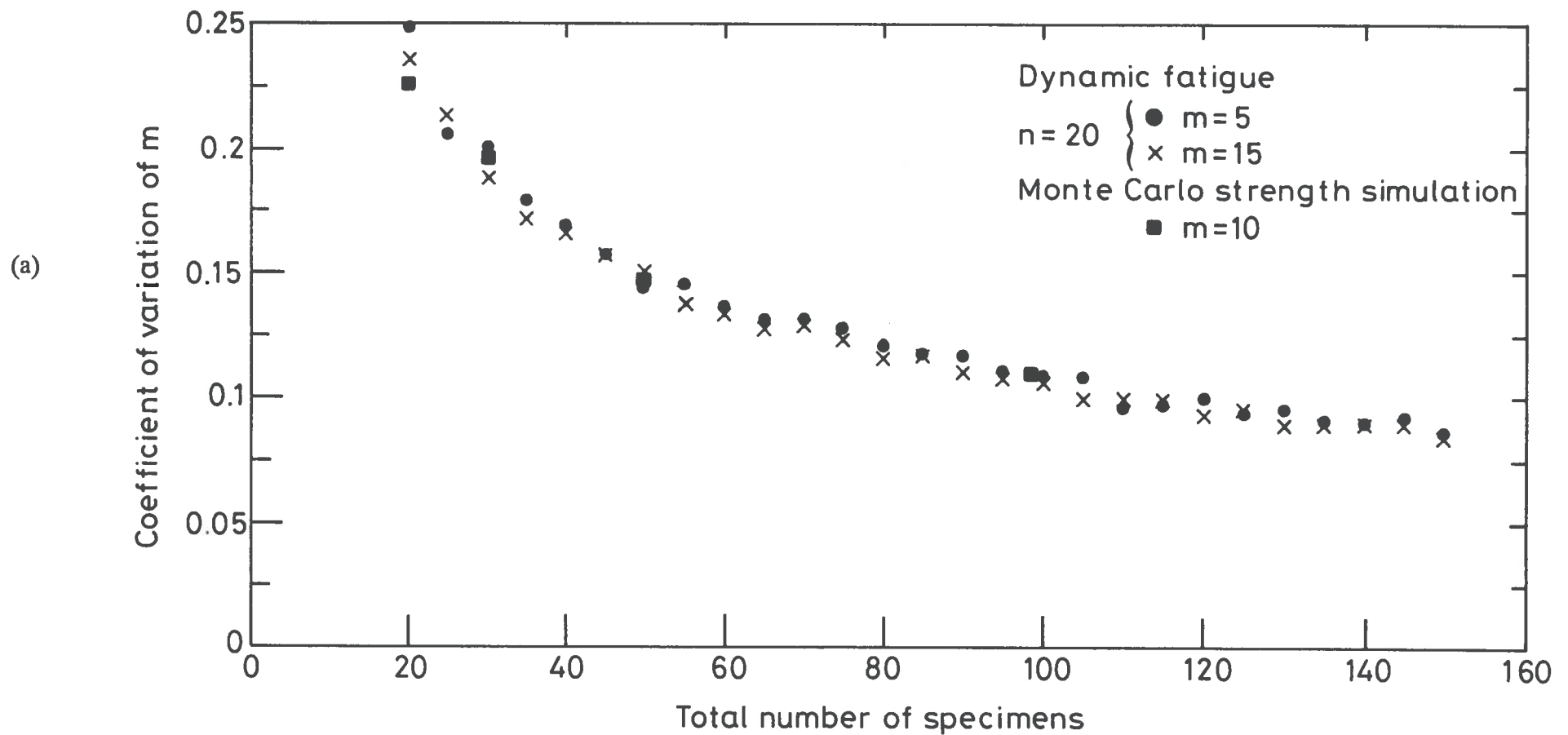


Figure 4. Coefficient of variation of two different values of m determined from analysis of computationally generated dynamic fatigue data, using 1,000 iterations to obtain each data point for (a) $n = 20$, and (b) $n = 100$. Data have also been included in (a) to show the results of analysis of computationally generated conventional strength data. The coefficient of variation is independent of m within statistical scatter.

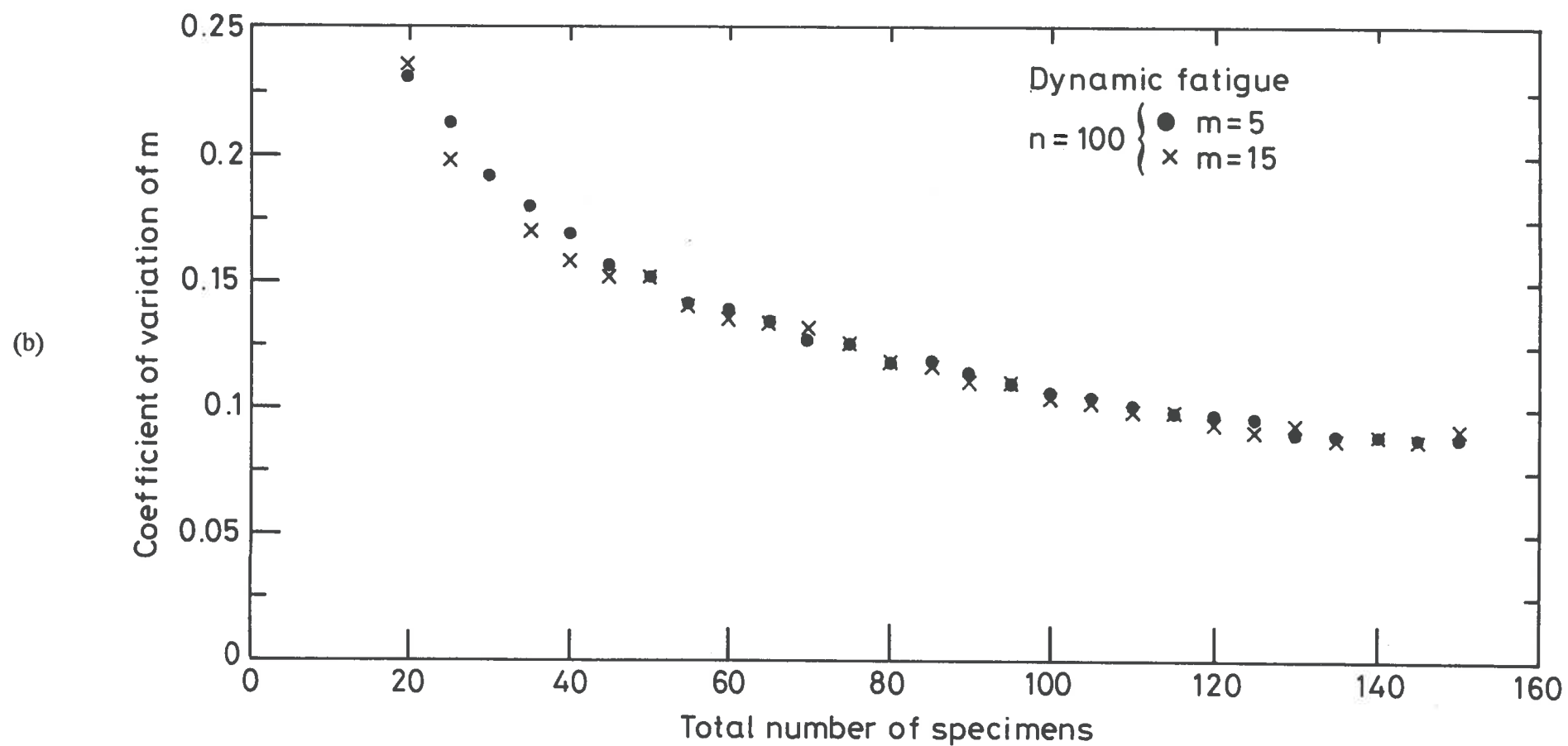


Figure 4(b).

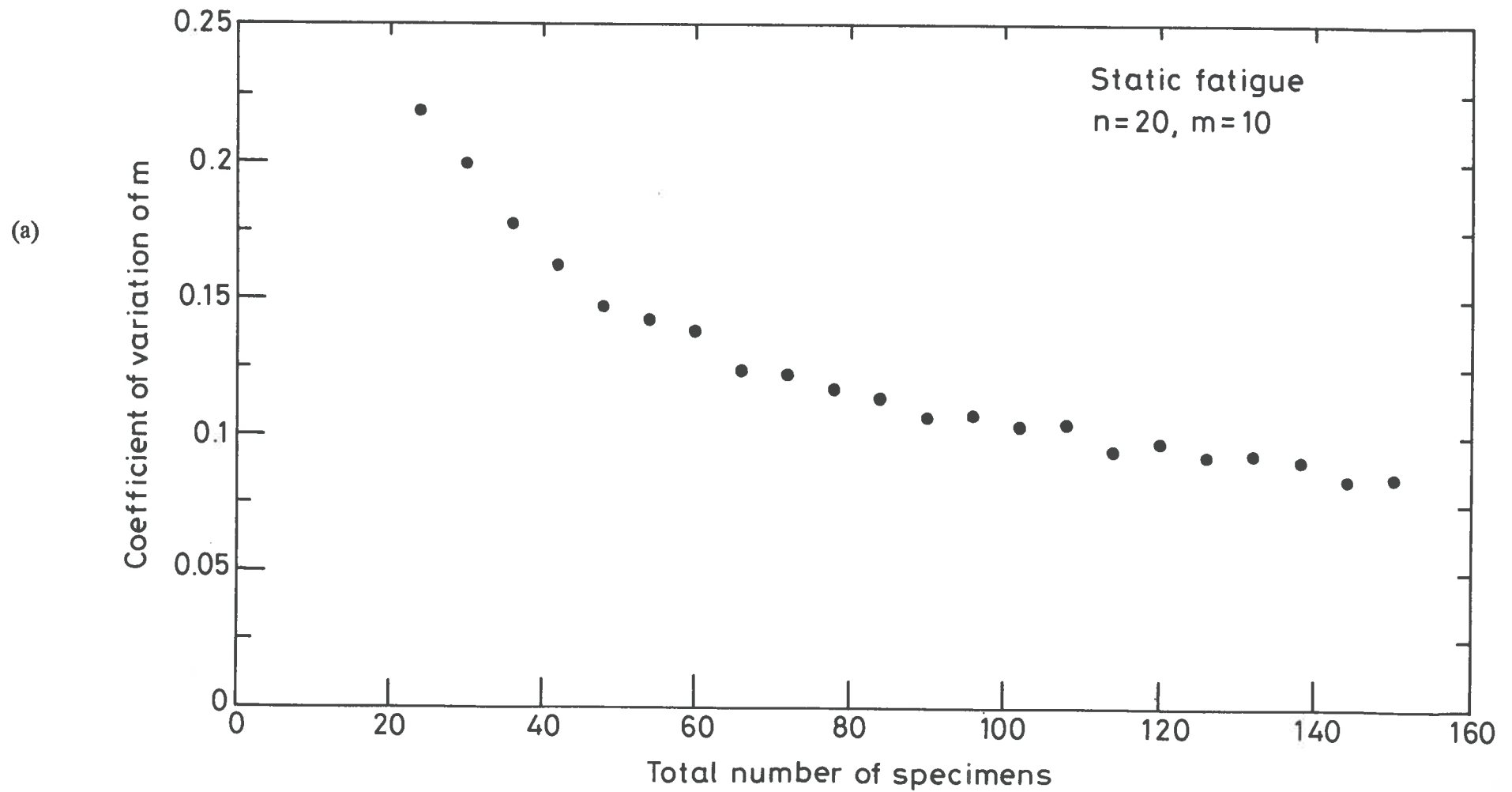


Figure 5. Coefficient of variation of m determined from analysis of computationally generated static fatigue data, using 1,000 iterations to obtain each data point, for (a) $m = 10, n = 20$, and (b) $m = 5$ and $15, n = 100$.

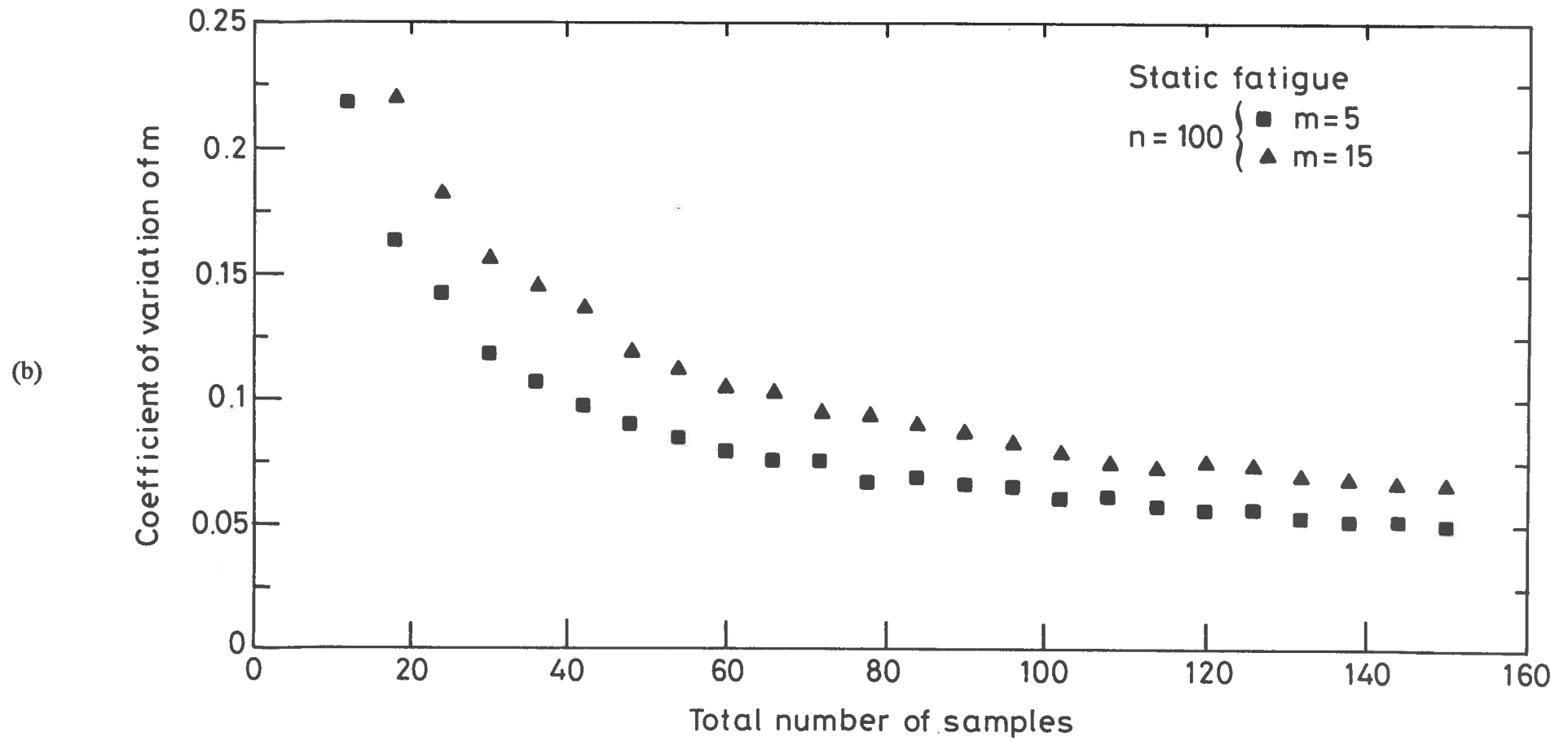


Figure 5(b).

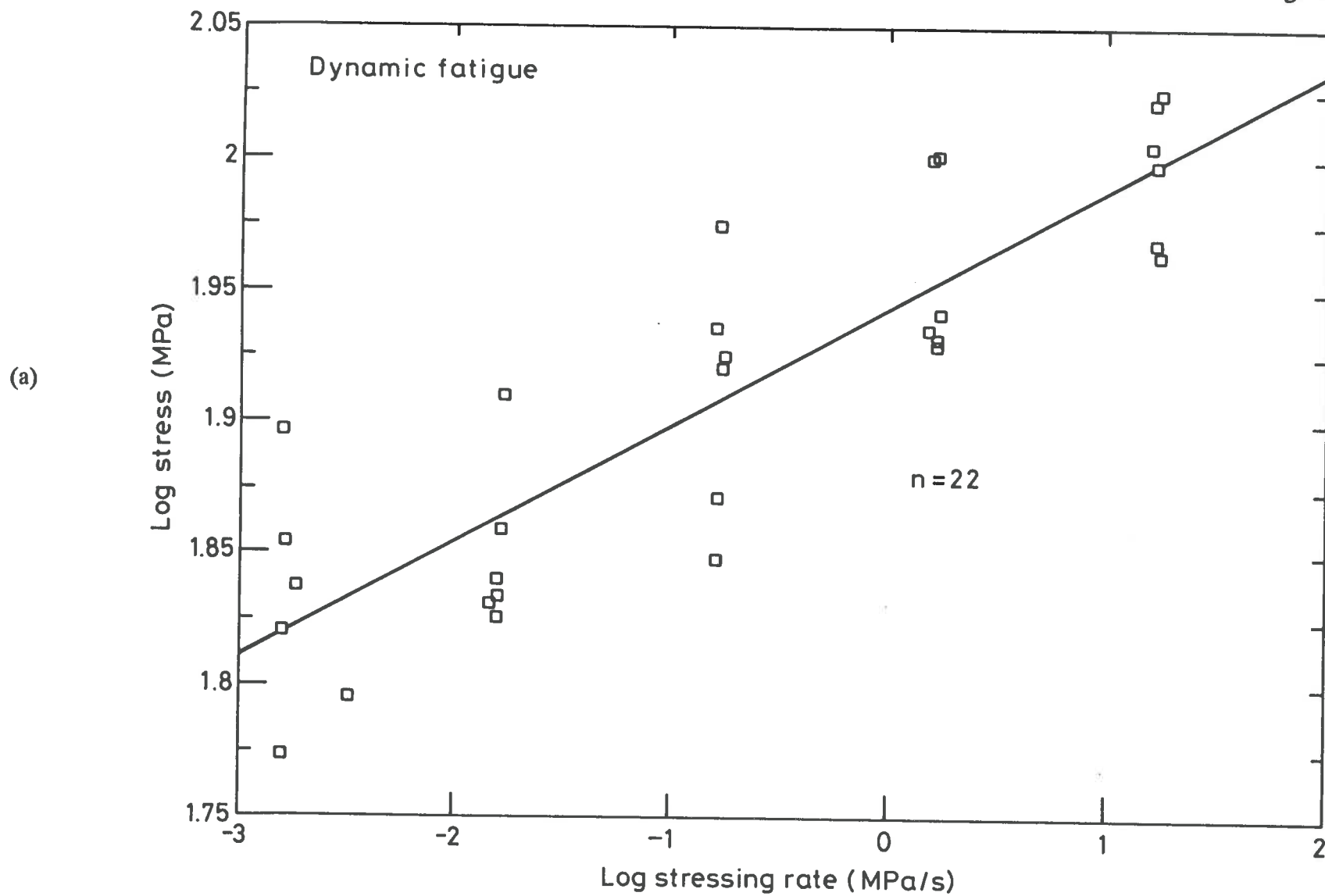


Figure 6. Dynamic fatigue data for soda-lime glass tested in distilled water: (a) raw data; (b) Weibull plot of data normalised to a single stressing rate using the value of n determined from (a).

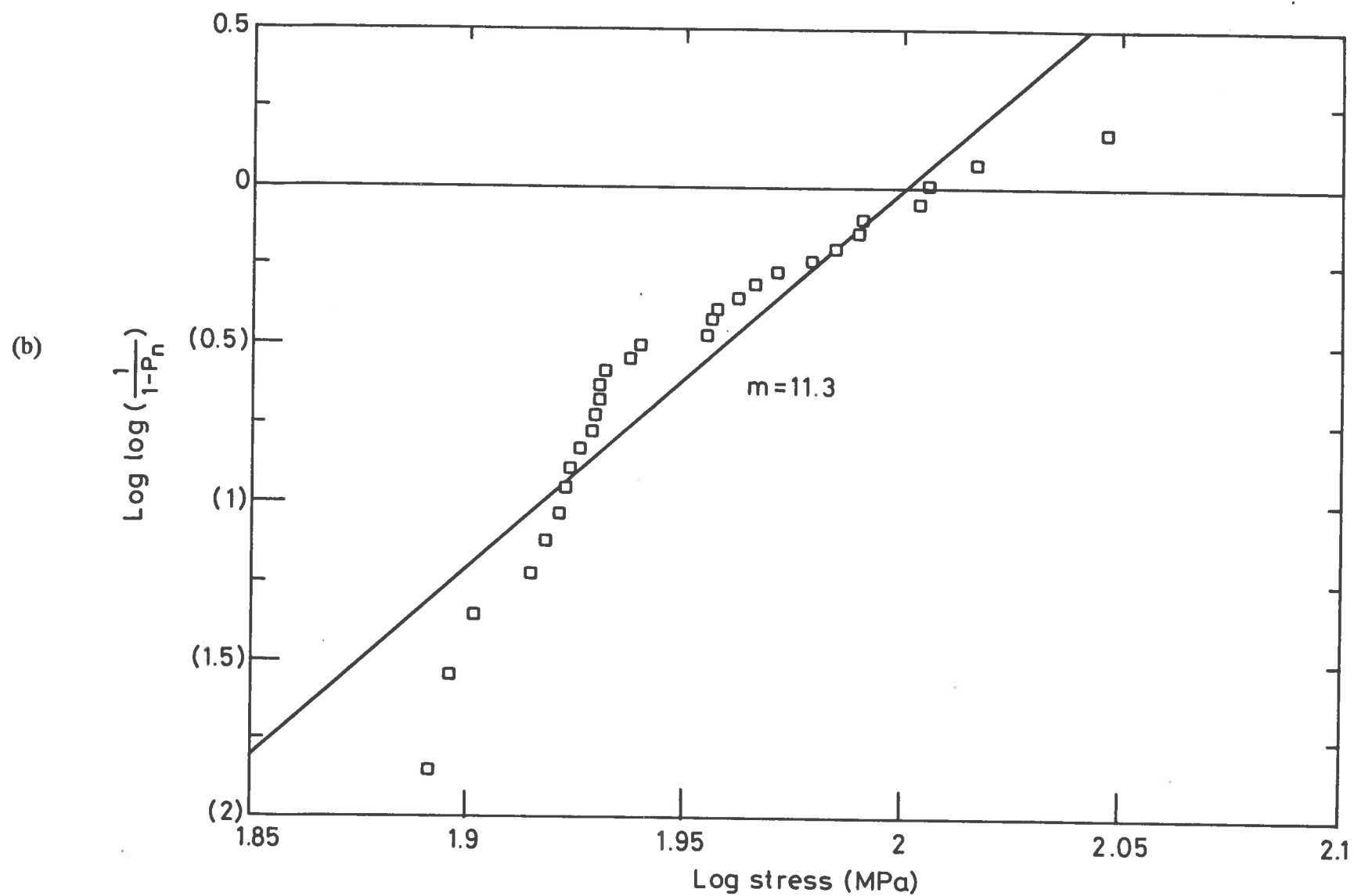


Figure 6(b).

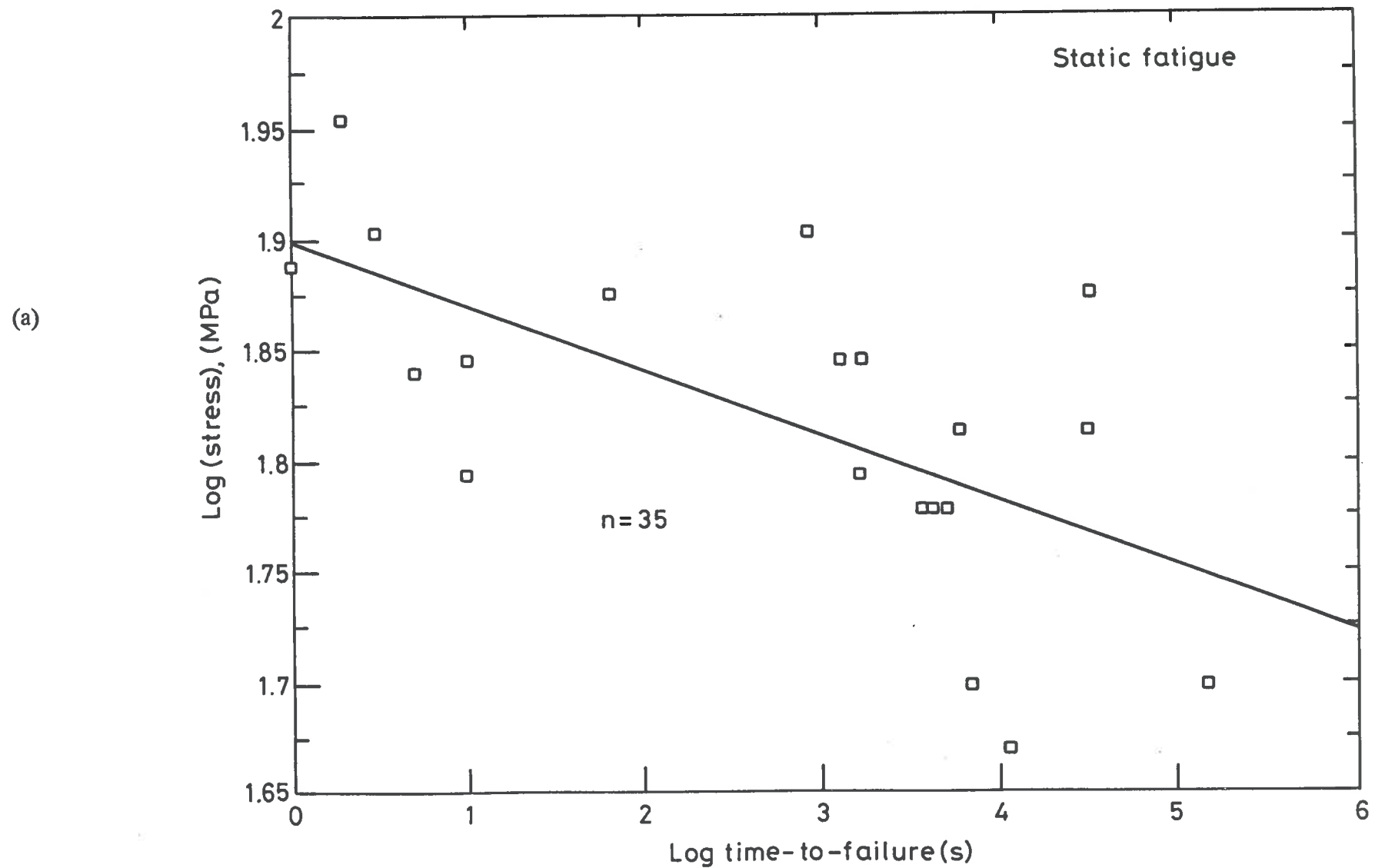


Figure 7. Static fatigue data for soda-lime glass tested in distilled water: (a) raw data; (b) Weibull plot of data normalised to a single time to failure using the value of n determined from (a).

(b)

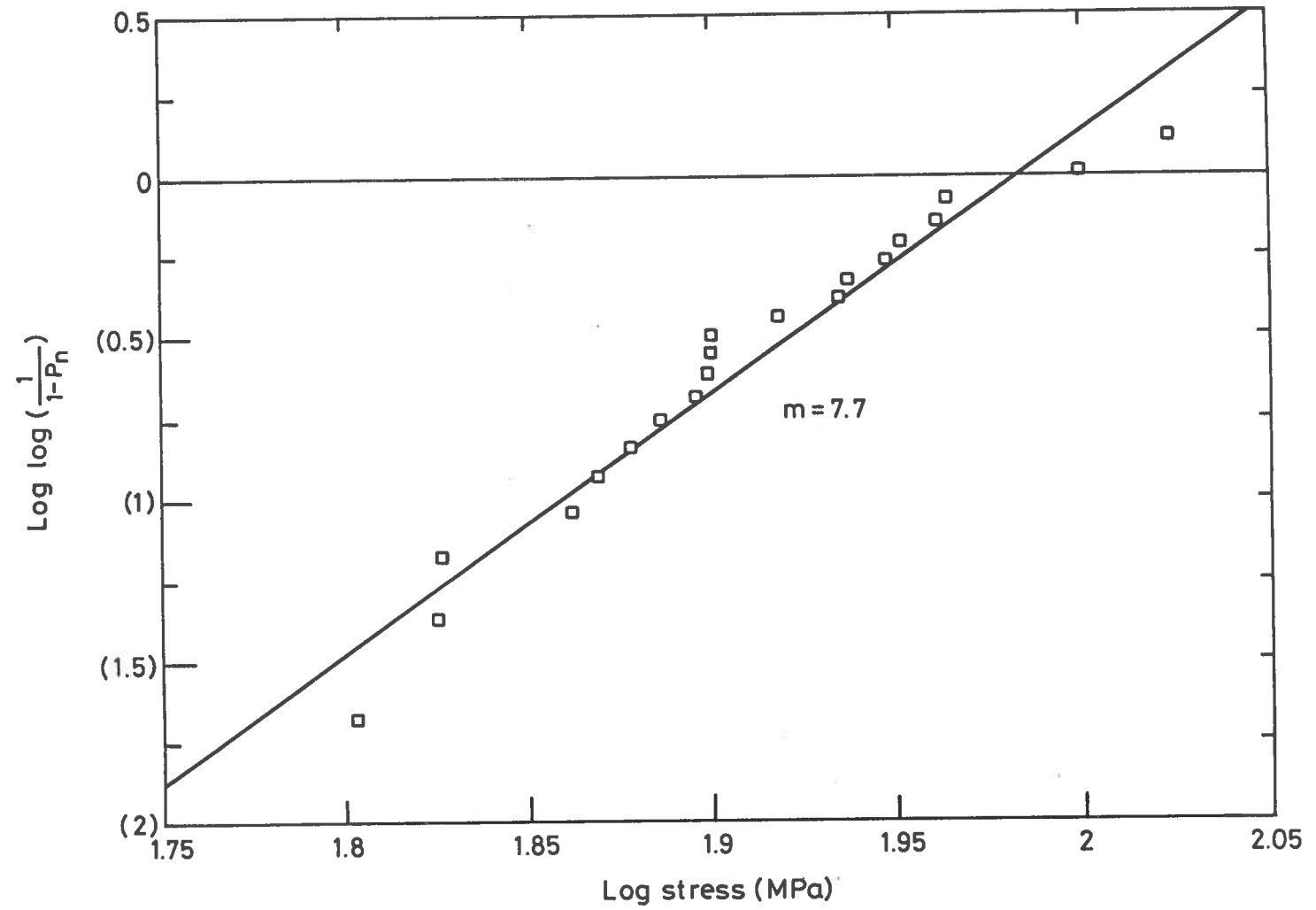


Figure 7(b).

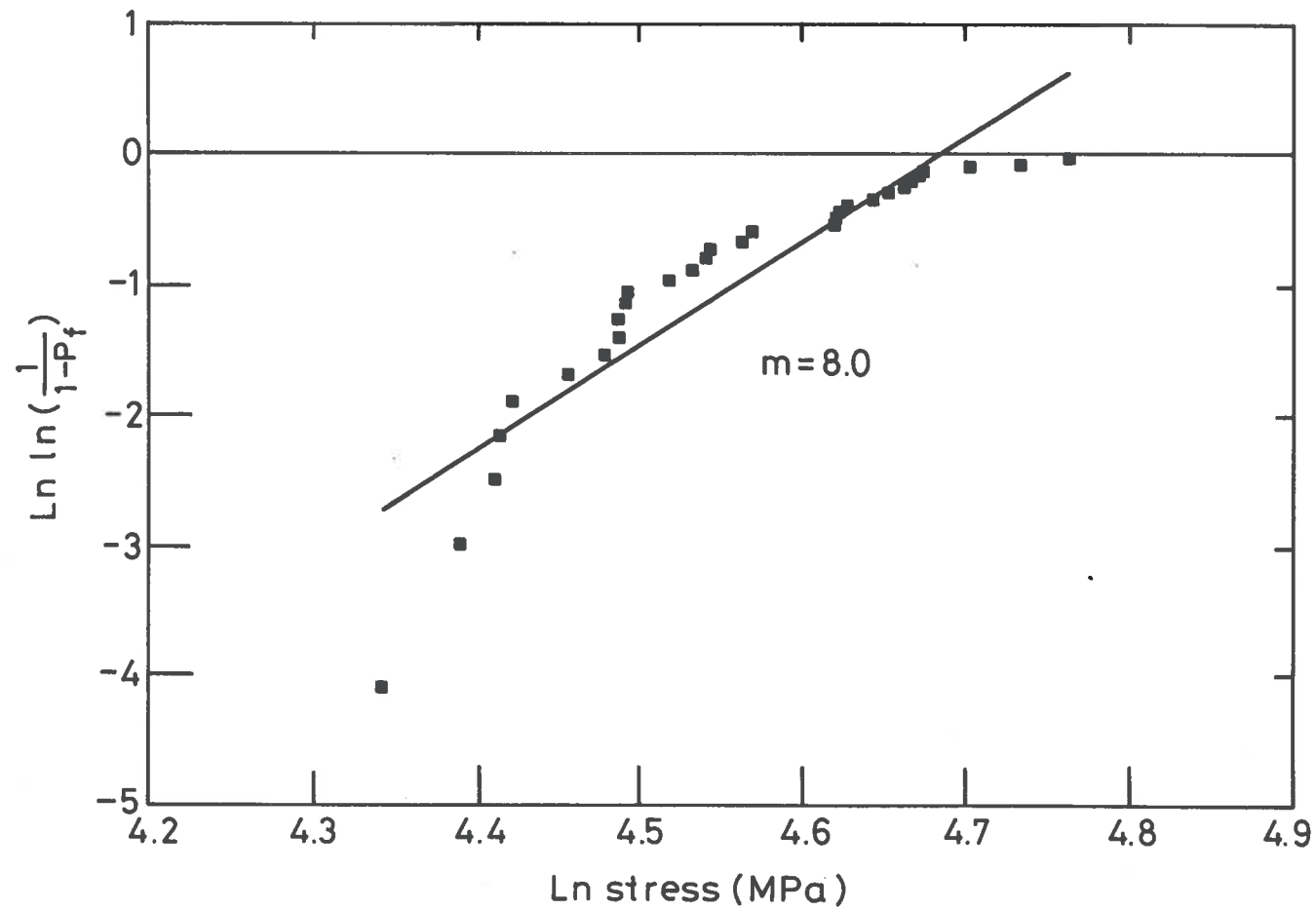


Figure 8. Weibull plot of strength data for soda-lime glass tested at a single stressing rate in distilled water.