



National Physical Laboratory

**Stress Transfer Mechanics for Multiple Perfectly Bonded
Concentric Cylinder Models of Unidirectional Composites**

L N McCartney

October 1993

Stress Transfer Mechanics for Multiple Perfectly Bonded Concentric Cylinder Models of Unidirectional Composites

L N McCartney
Division of Materials Metrology
National Physical Laboratory
Teddington
Middlesex TW11 0LW
United Kingdom

ABSTRACT

The problem of predicting the stress and displacement distributions in an assembly of perfectly bonded multiple cylinders subject to load transfer has been reduced to manipulating sets of recurrence relations that generate a set of fourth order simultaneous ordinary differential equations. Each cylinder is assumed to be a transverse isotropic solid and account is taken of the residual stresses arising from thermal expansion mismatch effects. Numerical solutions of the problem are readily obtained using a desk-top computer and experience of application of the technique to unidirectional carbon fibre epoxy composites has shown that the new technique is robust.

For a unidirectional composite with fibre fractures, by representing the fibres by one set of cylinders, the matrix by another set, and surrounding effective composite material by a third set, accurate solutions for stress and displacement distributions can be found in the vicinity of a fibre fracture that agree closely with finite element solutions. For the fibre fracture stress transfer problem, singularities are encountered for the axial stress in the matrix in the plane of the fibre fracture. These singularities are modelled reasonably well by the new technique whereas corresponding FEA predictions are subject to error in such regions. Stress transfer between fibre and matrix is affected significantly if the representative volume for the unidirectional composite is embedded in effective composite material. Stress transfer then occurs over a much larger distance along the fibre.

By smoothing out nearest neighbour fibres into the form of a cylindrical tube or shell, the multiple cylinder model can be used to estimate the stress concentrations that are induced in neighbouring fibres when an isolated fibre fracture occurs in a unidirectional composite. The value of the stress concentration is found to be slightly higher than that predicted by local load sharing if account is taken of the axial stress gradient in the nearest neighbour fibres, and smaller if the stress concentration is based on the average axial stress in these fibres.

© Crown copyright 1993

ISSN 0959-2423

National Physical Laboratory
Teddington, Middlesex TW11 0LW, UK

Extracts from this report may be reproduced provided the source is acknowledged.

Approved on behalf of Chief Executive, NPL,
by Dr M K Hossain, Head, Division of Materials Metrology

CONTENTS

	Page
1 INTRODUCTION	1
2 GEOMETRY OF THE ASSEMBLY OF CONCENTRIC CYLINDERS	1
3 EQUILIBRIUM EQUATIONS	1
4 STRESS-STRAIN-DISPLACEMENT RELATIONS	2
5 INTERFACIAL AND EXTERNAL RADIAL BOUNDARY CONDITIONS ...	2
6 ASSUMPTIONS OF THE RECURSIVE SOLUTION TECHNIQUE	3
7 DERIVATION OF OTHER BASIC RELATIONS (REGIONS: $r_{i-1} \leq r \leq r_i$, $i = 2 \dots N$)	4
8 DERIVATION OF OTHER BASIC RELATIONS (REGION: $0 \leq r \leq r_1$)	8
9 PARTITION OF THE SOLUTION	10
10 GENERATING THE LAMÉ SOLUTION	12
11 GENERATING THE PERTURBATION SOLUTION	14
12 DERIVATION OF SIMULTANEOUS DIFFERENTIAL EQUATIONS	15
13 CALCULATION OF THE COEFFICIENTS	16
14 AXIAL BOUNDARY CONDITIONS INVOLVING ONLY STRESSES	16
15 MIXED AXIAL BOUNDARY CONDITIONS	17
16 A SPECIAL CASE	20
17 STRESS TRANSFER ASSOCIATED WITH FIBRE FRACTURE	21
18 ESTIMATION OF STRESS CONCENTRATION FACTORS	23
19 CONCLUSIONS	25
REFERENCES	25
ACKNOWLEDGEMENTS	26
FIGURE CAPTIONS	28

1 INTRODUCTION

A multiple concentric cylinder model of a unidirectional composite has been used [1] to estimate the thermoelastic constants for the case when the composite does not contain any form of damage such as broken fibres. The objective of this paper is to show how stress transfer in a similar multiple concentric cylinder representation of a unidirectional composite can be estimated when fibre fractures occur. The new technique can be used to develop accurate solutions for stress and displacement distributions in two-phase composites by representing both fibre and matrix material by series of concentric cylinders having the properties of fibre or matrix material. The model can also be used to represent the effect of surrounding composite material on stress transfer associated with fibre fractures. The technique automatically allows the solution of stress transfer problems that arise in multi-phase composites. Such composites are encountered when coated fibres are incorporated into composite materials. Also reaction effects between fibre and matrix lead to the introduction of additional phases when metal matrix composites are operated at elevated temperatures. Multi-phase composites can be used to model non-uniform matrix properties associated with the transcrystalline layers that arise when carbon fibres are incorporated in epoxy matrix composites.

In this report a multiple concentric cylinder model of a unidirectional composite is analysed under biaxial tension, and subject to applied axial load distributions that give rise to axial stress transfer between cylinders through the action of distributed shear stresses on the interfaces between neighbouring cylinders. The model is constructed so that the concentric cylinders can be made of different transverse isotropic materials, so that they are perfectly bonded together, and so that the cylinder assembly is embedded in a surrounding transverse isotropic medium which has the same thermoelastic constants as the undamaged composite. The model takes full account of the effects of the thermal residual stresses induced in the composite during manufacture as a result of the difference in thermal expansion behaviour of fibre and matrix.

The solution technique is based on an earlier stress transfer model [2] that can deal only with two concentric cylinders. The extension to multiple cylinders involves the development of sets of recurrence relations used to generate simultaneous fourth order ordinary differential equations that can be solved by numerical methods, and to calculate the stress and displacement distributions throughout the cylinder assembly. Example stress and displacement distributions are presented to demonstrate that the solutions obtained by the new technique are in agreement with those obtained using finite element techniques [3].

2 GEOMETRY OF THE ASSEMBLY OF CONCENTRIC CYLINDERS

Assuming that the stress-transfer problems to be solved are axisymmetric, a set of cylindrical polar coordinates (r, θ, z) is introduced such that the origin lies on the axis of the innermost cylinder and the z -axis is directed along the axis of this cylinder. Fig.1 illustrates the geometry of the system where N cylinders are shown such that the i^{th} cylinder occupies the region $r_{i-1} \leq r \leq r_i$, for $i = 1 \dots N$, where $r_0 = 0$.

3 EQUILIBRIUM EQUATIONS

For axisymmetric problems the following equilibrium equations must be satisfied in every cylindrical element of the system

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \quad (1)$$

$$\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \sigma_{rz}}{\partial r} + \frac{\sigma_{rz}}{r} = 0. \quad (2)$$

4 STRESS-STRAIN-DISPLACEMENT RELATIONS

For each of the N cylindrical elements present, transverse isotropic stress-strain relations and corresponding strain-displacement relations are specified for $r_{i-1} \leq r \leq r_i$, $i = 1 \dots N$, by

$$\epsilon_{rr} \equiv \frac{\partial u_r}{\partial r} = \frac{\sigma_{rr}}{E_T^i} - \frac{\nu_T^i}{E_T^i} \sigma_{\theta\theta} - \frac{\nu_A^i}{E_A^i} \sigma_{zz} + \alpha_T^i \Delta T, \quad (3)$$

$$\epsilon_{\theta\theta} \equiv \frac{u_r}{r} = -\frac{\nu_T^i}{E_T^i} \sigma_{rr} + \frac{\sigma_{\theta\theta}}{E_T^i} - \frac{\nu_A^i}{E_A^i} \sigma_{zz} + \alpha_T^i \Delta T, \quad (4)$$

$$\epsilon_{zz} \equiv \frac{\partial u_z}{\partial z} = -\frac{\nu_A^i}{E_A^i} (\sigma_{rr} + \sigma_{\theta\theta}) + \frac{\sigma_{zz}}{E_A^i} + \alpha_A^i \Delta T, \quad (5)$$

$$2\epsilon_{rz} \equiv \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} = \frac{\sigma_{rz}}{\mu_A^i}, \quad (6)$$

where E_A^i , E_T^i , ν_A^i , ν_T^i , α_A^i and α_T^i are the axial and transverse Young's moduli, Poisson ratios and thermal expansion coefficients for the i^{th} cylindrical element. It should be noted that

$$E_T^i = 2\mu_T^i (1 + \nu_T^i) \text{ but } E_A^i \neq 2\mu_A^i (1 + \nu_A^i), \quad i = 1 \dots N, \quad (7)$$

where μ_A^i and μ_T^i are, respectively, the axial and transverse shear moduli of the i^{th} cylindrical element.

5 INTERFACIAL AND EXTERNAL RADIAL BOUNDARY CONDITIONS

At the interfaces $r = r_i$, $i = 1 \dots N-1$ between each cylindrical element of the system the following continuity conditions must be satisfied for all values of z

$$\sigma_r(r_i^-, z) = \sigma_r(r_i^+, z), \quad \sigma_{rz}(r_i^-, z) = \sigma_{rz}(r_i^+, z), \quad (8)$$

$$u_r(r_i^-, z) = u_r(r_i^+, z), \quad u_z(r_i^-, z) = u_z(r_i^+, z). \quad (9)$$

These conditions assume that there is perfect bonding between each cylindrical element.

On the external surface $r = r_N$ of the assembly of cylinders one of the following two sets of boundary conditions is imposed

$$\text{PROBLEM I: } \sigma_r(r_N, z) = \sigma_T, \quad \sigma_{rz}(r_N, z) = 0, \quad (10)$$

or

$$\text{PROBLEM II: } \sigma_{rz}(r_N, z) = 0, \quad u_r(r_N, z) = u, \quad (11)$$

where σ_T is a uniform transverse applied stress, and where u is a uniform radial displacement. These boundary conditions, when applied in conjunction with the generalised plane strain requirement of a uniform strain ϵ being imposed in the z -direction, lead to a problem whose solution corresponds to the multiple cylinder Lamé solution [1]. For many applications the parameter u will be specified by the value of u_r on the external boundary $r = r_N$ given by the Lamé solution of problem I. This situation is assumed in this report. There are many problems which arise where the strain ϵ in the z -direction is not uniform because of some form of axial stress transfer between the various cylindrical elements. The solution of such problems is the principal objective of this report. The solutions of problems I and II are expected to represent extremes of stress transfer behaviour; problem I having no lateral constraint while problem II maximises such constraint when $u = 0$.

6 ASSUMPTIONS OF THE RECURSIVE SOLUTION TECHNIQUE

The starting point for the development of the recursive solution technique for stress transfer problems is to *assume* the following representation for the shear stress component

For $0 \leq r \leq r_1$

$$\sigma_{rz} = C_1'(z) \frac{r}{r_1}. \quad (12)$$

For $r_{i-1} \leq r \leq r_i, i = 2 \dots N$

$$\sigma_{rz} = \lambda_i \left[C_{i-1}'(z) \left(\frac{r_i}{r} - \frac{r}{r_i} \right) - C_i'(z) \left(\frac{r_{i-1}}{r} - \frac{r}{r_{i-1}} \right) \right], \quad (13)$$

$$\text{where } \lambda_i = \frac{r_i r_{i-1}}{r_i^2 - r_{i-1}^2}, \quad i = 2 \dots N. \quad (14)$$

The stress component σ_{rz} is automatically continuous at $r = r_i, i = 1 \dots N-1$ thus satisfying the interfacial boundary condition (8)₂. The functions $C_i(z), i = 1 \dots N$ are unknowns which are to be determined. It follows from (12) and (13) that

$$\sigma_{rz}(r_i, z) \equiv C_i'(z), \quad i = 1 \dots N. \quad (15)$$

A corresponding representation for the radial displacement components is *assumed* of the form

For $0 \leq r \leq r_1$

$$u_r = u_1(z) \frac{r}{r_1}, \quad (16)$$

For $r_{i-1} \leq r \leq r_i$, $i = 2 \dots N$

$$u_r = \lambda_i \left[u_{i-1}(z) \left(\frac{r_i}{r} - \frac{r}{r_i} \right) - u_i(z) \left(\frac{r_{i-1}}{r} - \frac{r}{r_{i-1}} \right) \right], \quad (17)$$

so that the displacement component u_r is continuous at $r = r_i$, $i = 1 \dots N-1$ thus satisfying the interfacial boundary condition (9). It follows from (16) and (17) that

$$u_r(r_i, z) \equiv u_i(z), \quad i = 1 \dots N. \quad (18)$$

The functions $u_i(z)$, $i = 1 \dots N$ are also unknowns to be determined by the recursive technique.

7 DERIVATION OF OTHER BASIC RELATIONS (REGIONS: $r_{i-1} \leq r \leq r_i$, $i = 2 \dots N$)

On subtracting the relations (3) and (4)

$$\varepsilon_{rr} - \varepsilon_{\theta\theta} = \frac{1}{2\mu_i} (\sigma_{rr} - \sigma_{\theta\theta}) = r \frac{\partial}{\partial r} \left(\frac{u_r}{r} \right). \quad (19)$$

On substituting (17) in (19) it can be shown that

$$\sigma_{rr} - \sigma_{\theta\theta} = 4\mu_i \lambda_i \frac{r_{i-1} u_i(z) - r_i u_{i-1}(z)}{r^2}. \quad (20)$$

The substitution of (13) and (20) in the equilibrium equation (1) leads to

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} = & \lambda_i \left[C_i''(z) \left(\frac{r_{i-1}}{r} - \frac{r}{r_{i-1}} \right) - C_{i-1}''(z) \left(\frac{r_i}{r} - \frac{r}{r_i} \right) \right] \\ & + 4\mu_i \lambda_i \frac{r_i u_{i-1}(z) - r_{i-1} u_i(z)}{r^3}. \end{aligned} \quad (21)$$

Integration with respect to r then yields

$$\begin{aligned} \sigma_{rr} = & \lambda_i \left[C_i''(z) \left(r_{i-1} \ln \frac{r}{r_i} - \frac{r^2}{2r_{i-1}} + \frac{r_i^2}{2r_{i-1}} \right) - C_{i-1}''(z) \left(r_i \ln \frac{r}{r_i} - \frac{r^2}{2r_i} + \frac{r_i}{2} \right) \right] \\ & + 2\mu_i \lambda_i \left[r_{i-1} u_i(z) - r_i u_{i-1}(z) \right] \left(\frac{1}{r^2} - \frac{1}{r_i^2} \right) + \sigma_r^i(z), \end{aligned} \quad (22)$$

where $\sigma_r^i(z)$ are the radial stresses acting on the interfaces $r = r_i$, $i = 2 \dots N-1$ and on the external surface $r = r_N$. The stress component $\sigma_{\theta\theta}$ can then be calculated using (20). Since (8),

asserts that σ_r must be continuous at $r = r_i$, $i = 1 \dots N-1$ it follows from (22) that a recurrence relation can be derived of the form

$$\begin{aligned} \sigma_r^{i-1}(z) &= \sigma_r^i(z) + \left[\lambda_i r_{i-1} \ln \frac{r_{i-1}}{r_i} + \frac{1}{2} r_i \right] C_i''(z) \\ &- \left[\lambda_i r_i \ln \frac{r_{i-1}}{r_i} + \frac{1}{2} r_{i-1} \right] C_{i-1}''(z) + 2\mu_i \left[\frac{u_i(z)}{r_i} - \frac{u_{i-1}(z)}{r_{i-1}} \right], \quad i = N \dots 2. \end{aligned} \quad (23)$$

On using (2) and (13)

$$\frac{\partial \sigma_{zz}}{\partial z} = 2\lambda_i \left(\frac{C_{i-1}'(z)}{r_i} - \frac{C_i'(z)}{r_{i-1}} \right), \quad (24)$$

and on integration with respect to z

$$\sigma_{zz} = \sigma_i + 2\lambda_i \left(\frac{C_{i-1}(z)}{r_i} - \frac{C_i(z)}{r_{i-1}} \right), \quad (25)$$

where σ_i is assumed to be independent of r . Since σ_{zz} is a constant in each cylindrical element for the Lamé solution it will become clear later that this assumption does not introduce an additional approximation.

On using (6), (13) and (17)

$$\frac{\partial u_z}{\partial r} = \lambda_i \left[\frac{C_{i-1}'(z)}{\mu_A} - u_{i-1}'(z) \right] \left(\frac{r_i}{r} - \frac{r}{r_i} \right) - \lambda_i \left[\frac{C_i'(z)}{\mu_A} - u_i'(z) \right] \left(\frac{r_{i-1}}{r} - \frac{r}{r_{i-1}} \right). \quad (26)$$

Integration with respect to r then leads to

$$\begin{aligned} u_z &= \lambda_i \left(\frac{C_{i-1}'(z)}{\mu_A} - u_{i-1}'(z) \right) \left[r_i \ln \frac{r}{r_i} - \frac{r^2}{2r_i} + \frac{r_i}{2} \right] \\ &- \lambda_i \left(\frac{C_i'(z)}{\mu_A} - u_i'(z) \right) \left[r_{i-1} \ln \frac{r}{r_i} - \frac{r^2}{2r_{i-1}} + \frac{r_i^2}{2r_{i-1}} \right] + H_i(z), \end{aligned} \quad (27)$$

where $H_i(z)$, $i = 2 \dots N$ are functions which have to be determined as part of the solution. The integration of (27) has clearly been carried out so that

$$H_i(z) \equiv u_z(r_i, z), \quad i = 2 \dots N. \quad (28)$$

From (9)₂ the continuity of u_z at $r = r_i$, $i = 2 \dots N-1$ leads to

$$\begin{aligned}
H_{i-1}(z) = H_i(z) + \left(\frac{C'_{i-1}(z)}{\mu_A^i} - u'_{i-1}(z) \right) \left[\lambda_i r_i \ln \frac{r_{i-1}}{r_i} + \frac{1}{2} r_{i-1} \right] \\
- \left(\frac{C'_i(z)}{\mu_A^i} - u'_i(z) \right) \left[\lambda_i r_{i-1} \ln \frac{r_{i-1}}{r_i} + \frac{r_i}{2} \right], \quad i = 2 \dots N,
\end{aligned} \quad (29)$$

where $H_i(z) \equiv u_z(r_i, z)$.

On adding the relations (3) and (4)

$$E_T^i \frac{1}{r} \frac{\partial}{\partial r} (ru_r) = (1 - \nu_T^i) (\sigma_{rr} + \sigma_{\theta\theta}) - 2\nu_A^i \frac{E_T^i}{E_A^i} \sigma_{zz} + 2E_T^i \alpha_T^i \Delta T. \quad (30)$$

On averaging (30) over the section of the i^{th} cylindrical element using the notation (for $i = 1 \dots N$)

$$\bar{f}(z) = \frac{1}{\pi(r_i^2 - r_{i-1}^2)} \int_{r_{i-1}}^{r_i} 2\pi r f(r, z) dr = \frac{2\lambda_i}{r_i r_{i-1}} \int_{r_{i-1}}^{r_i} r f(r, z) dr, \quad (31)$$

it follows from (17) that for $i = 2 \dots N$

$$2\lambda_i E_T^i \left(\frac{u_i(z)}{r_{i-1}} - \frac{u_{i-1}(z)}{r_i} \right) = (1 - \nu_T^i) (\bar{\sigma}_{rr} + \bar{\sigma}_{\theta\theta}) - 2\nu_A^i \frac{E_T^i}{E_A^i} \sigma_{zz} + 2E_T^i \alpha_T^i \Delta T. \quad (32)$$

Now $\sigma_{rr} + \sigma_{\theta\theta} = 2\sigma_r - (\sigma_r - \sigma_{\theta\theta})$ so that from (20) and (22)

$$\begin{aligned}
\sigma_{rr} + \sigma_{\theta\theta} = 2\lambda_i \left[C_i''(z) \left(r_{i-1} \ln \frac{r}{r_i} - \frac{r^2}{2r_{i-1}} + \frac{r_i^2}{2r_{i-1}} \right) - C_{i-1}''(z) \left(r_i \ln \frac{r}{r_i} - \frac{r^2}{2r_i} + \frac{r_i}{2} \right) \right] \\
- \frac{4\mu_T^i \lambda_i}{r_i^2} [r_{i-1} u_i(z) - r_i u_{i-1}(z)] + 2\sigma_T^i(z).
\end{aligned} \quad (33)$$

Define (for $i = 2 \dots N$)

$$\bar{F}_1^i = \frac{2\lambda_i}{r_i r_{i-1}} \int_{r_{i-1}}^{r_i} \left(r_{i-1} r \ln \frac{r}{r_i} - \frac{r^3}{2r_{i-1}} + \frac{r_i^2 r}{2r_{i-1}} \right) dr = \left(\frac{r_i^2 - 3r_{i-1}^2}{4r_{i-1}} - \frac{\lambda_i r_{i-1}^2}{r_i} \ln \frac{r_{i-1}}{r_i} \right), \quad (34)$$

and

$$\bar{F}_2^i = \frac{2\lambda_i}{r_i r_{i-1}} \int_{r_{i-1}}^{r_i} \left(r_i r \ln \frac{r}{r_i} - \frac{r^3}{2r_i} + \frac{r_i r}{2} \right) dr = - \left(\frac{r_i^2 + r_{i-1}^2}{4r_i} + \lambda_i r_{i-1} \ln \frac{r_{i-1}}{r_i} \right). \quad (35)$$

Thus on averaging (33) for $i = 2 \dots N$

$$\begin{aligned} \bar{\sigma}_{rr} + \bar{\sigma}_{\theta\theta} &= 2\lambda_i \left[\bar{F}_1^i C_i''(z) - \bar{F}_2^i C_{i-1}''(z) \right] \\ &- \frac{4\mu_T^i \lambda_i}{r_i^2} \left[r_{i-1} u_i(z) - r_i u_{i-1}(z) \right] + 2\sigma_T^i(z), \end{aligned} \quad (36)$$

and substitution with (25) in (32) leads to

$$\begin{aligned} &\frac{\lambda_i}{r_{i-1}} \left[1 + \frac{1-v_T^i}{1+v_T^i} \frac{r_{i-1}^2}{r_i^2} \right] u_i(z) - \frac{\lambda_i}{r_i} \left[1 + \frac{1-v_T^i}{1+v_T^i} \right] u_{i-1}(z) \\ &- \lambda_i \frac{1-v_T^i}{E_T^i} \left[\bar{F}_1^i C_i''(z) - \bar{F}_2^i C_{i-1}''(z) \right] + \frac{2\lambda_i v_A^i}{E_A^i} \left[\frac{C_{i-1}(z)}{r_i} - \frac{C_i(z)}{r_{i-1}} \right] \\ &= \frac{1-v_T^i}{E_T^i} \sigma_T^i(z) - \frac{v_A^i}{E_A^i} \sigma_i + \alpha_T^i \Delta T, \quad i = 2 \dots N. \end{aligned} \quad (37)$$

On averaging the stress-strain relation (5) using (31) it follows that

$$\frac{d\bar{u}_z}{dz} = - \frac{v_A^i}{E_A^i} (\bar{\sigma}_{rr} + \bar{\sigma}_{\theta\theta}) + \frac{\sigma_{zz}}{E_A^i} + \alpha_A^i \Delta T. \quad (38)$$

From (27), (31), (34) and (35), for $i = 2 \dots N$,

$$\bar{u}_z = \bar{u}_z^i(z) \equiv \lambda_i \left(\frac{C_{i-1}'(z)}{\mu_A^i} - u_{i-1}'(z) \right) \bar{F}_2^i - \lambda_i \left(\frac{C_i'(z)}{\mu_A^i} - u_i'(z) \right) \bar{F}_1^i + H_i(z). \quad (39)$$

Thus on substituting (25), (36) and (39) in (38) it can be shown that

$$\begin{aligned}
& \lambda_i \left(\frac{1}{\mu_A^i} - \frac{2\nu_A^i}{E_A^i} \right) \left[\bar{F}_2^i C_{i-1}''(z) - \bar{F}_1^i C_i''(z) \right] - \lambda_i \left(\bar{F}_2^i u_{i-1}''(z) - \bar{F}_1^i u_i''(z) \right) \\
& - \frac{2\lambda_i}{E_A^i} \left(\frac{C_{i-1}(z)}{r_i} - \frac{C_i(z)}{r_{i-1}} \right) + \frac{4\lambda_i \mu_T^i \nu_A^i}{E_A^i r_i^2} \left(r_i u_{i-1}(z) - r_{i-1} u_i(z) \right) \\
& = - \frac{2\nu_A^i}{E_A^i} \sigma_T^i(z) + \frac{\sigma_i}{E_A^i} + \alpha_A^i \Delta T - H_i'(z), \quad i = 2 \dots N.
\end{aligned} \tag{40}$$

8 DERIVATION OF OTHER BASIC RELATIONS (REGION: $0 \leq r \leq r_1$)

It follows from (3), (4), and (16) that

$$\varepsilon_{\theta\theta} = \varepsilon_{rr} \quad \text{and} \quad \sigma_{\theta\theta} = \sigma_{rr} \tag{41}$$

so that on using the equilibrium equation (1) with (12)

$$\frac{\partial \sigma_{rr}}{\partial r} = - C_1''(z) \frac{r}{r_1}. \tag{42}$$

Integration with respect to r then yields

$$\sigma_{rr} \equiv \sigma_{\theta\theta} = C_1''(z) \frac{r_1^2 - r^2}{2r_1} + \sigma_T^1(z), \tag{43}$$

where $\sigma_T^1(z)$ is the transverse stress acting on the interface $r = r_1$. Clearly the continuity relation (8)₁ is satisfied on this interface.

The use of the equilibrium equation (2) together with the relation (12) leads to

$$\frac{\partial \sigma_{zz}}{\partial z} = - 2 C_1'(z) \frac{1}{r_1}, \tag{44}$$

so that on integration with respect to z

$$\sigma_{zz} = \sigma_1 - \frac{2}{r_1} C_1(z), \tag{45}$$

where σ_1 is *assumed* to be independent of r , consistent with the assumption made when deriving (25).

On using (6), (12) and (16)

$$\frac{\partial u_z}{\partial r} = -u_1'(z) \frac{r}{r_1} + \frac{C_1'(z)}{\mu_A^1} \frac{r}{r_1} . \quad (46)$$

Integration with respect to r yields

$$u_z = \left[\frac{C_1'(z)}{\mu_A^1} - u_1'(z) \right] \frac{r^2 - r_1^2}{2r_1} + H_1(z) , \quad (47)$$

where $H_1(z)$ is a function which must be determined. The integration of (46) has been carried out so that

$$H_1(z) \equiv u_z(r_1, z) , \quad (48)$$

ensuring that the continuity condition (9)₂ is satisfied at $r = r_1$.

On adding (3) and (4) the relation (30) is obtained for $i = 1$, namely

$$E_T^1 \frac{1}{r} \frac{\partial}{\partial r} (ru_r) = (1 - \nu_T^1) (\sigma_{rr} + \sigma_{\theta\theta}) - 2\nu_A^1 \frac{E_T^1}{E_A^1} \sigma_{zz} + 2E_T^1 \alpha_T^1 \Delta T , \quad (49)$$

and on averaging using the notation

$$\bar{f}(z) = \frac{1}{\pi r_1^2} \int_0^{r_1} 2\pi r f(r, z) dr , \quad (50)$$

it follows from (16) and (41)₂ that

$$E_T^1 \frac{u_1(z)}{r_1} = (1 - \nu_T^1) \bar{\sigma}_{rr} - \nu_A^1 \frac{E_T^1}{E_A^1} \sigma_{zz} + E_T^1 \alpha_T^1 \Delta T . \quad (51)$$

Now on averaging (43)

$$\bar{\sigma}_{rr} = \frac{2}{r_1^2} \int_0^{r_1} \left[C_1''(z) \frac{r r_1^2 - r^3}{2r_1} + \sigma_T^1(z) r \right] dr = \frac{1}{4} r_1 C_1''(z) + \sigma_T^1(z) . \quad (52)$$

Therefore, on substituting (45) and (52) in (51)

$$\frac{u_1(z)}{r_1} - \frac{1 - \nu_T^1}{4E_T^1} r_1 C_1''(z) - \frac{2\nu_A^1}{E_A^1} \frac{C_1(z)}{r_1} = \frac{1 - \nu_T^1}{E_T^1} \sigma_T^1(z) - \frac{\nu_A^1}{E_A^1} \sigma_1 + \alpha_T^1 \Delta T . \quad (53)$$

It follows from (47) and (50) that

$$\bar{u}_z = \bar{u}_z^1(z) \equiv -\frac{r_1}{4} \left(\frac{C_1'(z)}{\mu_A^1} - u_1'(z) \right) + H_1(z). \quad (54)$$

Therefore, on averaging (5) using (45), (52) and (54) it can be shown that

$$\begin{aligned} -\frac{r_1}{4} \left(\frac{1}{\mu_A^1} - \frac{2\nu_A^1}{E_A^1} \right) C_1''(z) + \frac{r_1}{4} u_1''(z) + \frac{2}{r_1 E_A^1} C_1(z) \\ = -\frac{2\nu_A^1}{E_A^1} \sigma_T^1(z) + \frac{\sigma_1}{E_A^1} + \alpha_A^1 \Delta T - H_1'(z). \end{aligned} \quad (55)$$

9 PARTITION OF THE SOLUTION

It is useful to partition the interfacial values of the radial stress σ_r and the radial displacement u_r as follows

$$\left. \begin{aligned} \sigma_r^i(z) &= \Delta\sigma_r^i(z) + \sigma_r^i \\ u_r^i(z) &= \Delta u_r^i(z) + u_r^i \end{aligned} \right\} i = 1 \dots N \quad (56)$$

where σ_r^i and u_r^i , $i = 1 \dots N$ are the values of σ_r^i and $u_r^i(z)$ for the Lamé solution corresponding to the case when there is no axial stress-transfer and the uniform strain in the axial direction corresponding to the applied stress σ is given by ϵ . For this case the axial stress σ_{zz} in the i^{th} cylinder has a constant value σ_r^i , and the functions $C_i(z)$, $\Delta\sigma_r^i(z)$ and $\Delta u_r^i(z)$, $i = 1 \dots N$, are all identically zero. This explains why no additional approximation is introduced in (25) and (45) when the parameters σ_r^i , $i = 1 \dots N$ are assumed to be independent of r . The functions $C_i(z)$, $\Delta\sigma_r^i(z)$ and $\Delta u_r^i(z)$, $i = 1 \dots N$, denote that part of the solution representing the perturbation from the Lamé solution and characterising the degree of stress transfer. The relation (23) may then be written in the form of a recurrence relation generating values of the perturbations $\Delta\sigma_r^i(z)$, $i = N-1 \dots 1$

$$\begin{aligned} \Delta\sigma_r^{i-1}(z) &= \Delta\sigma_r^i(z) + \alpha_i C_i''(z) - \beta_i C_{i-1}''(z) \\ &\quad + 2\mu_i^1 \left(\frac{\Delta u_r^i(z)}{r_i} - \frac{\Delta u_{r-1}^i(z)}{r_{i-1}} \right), \quad i = N \dots 2, \end{aligned} \quad (57)$$

where

$$\left. \begin{aligned} \alpha_i &= \lambda_i r_{i-1} \ln \frac{r_{i-1}}{r_i} + \frac{1}{2} r_i, \\ \beta_i &= \lambda_i r_i \ln \frac{r_{i-1}}{r_i} + \frac{1}{2} r_{i-1}, \end{aligned} \right\} i = 2 \dots N. \quad (58)$$

In addition the corresponding Lamé part of the partition is such that

$$\sigma_T^{i-1} = \sigma_T^i + 2\mu_T^i \left(\frac{u_i}{r_i} - \frac{u_{i-1}}{r_{i-1}} \right), \quad i = N \dots 2. \quad (59)$$

On using (56) the relation (37) may be written

$$\begin{aligned} \left(\frac{r_i}{r_{i-1}} + \frac{1-\nu_T^i}{1+\nu_T^i} \frac{r_{i-1}}{r_i} \right) \frac{\Delta u_i(z)}{r_i} - \frac{r_{i-1}}{r_i} \frac{2}{1+\nu_T^i} \frac{\Delta u_{i-1}(z)}{r_{i-1}} = \\ \frac{1-\nu_T^i}{E_T^i} \left(\bar{F}_1^i C_i''(z) - \bar{F}_2^i C_{i-1}''(z) \right) - \frac{2\nu_A^i}{E_A^i} \left(\frac{C_{i-1}(z)}{r_i} - \frac{C_i(z)}{r_{i-1}} \right) \\ + \frac{1-\nu_T^i}{E_T^i} \frac{\Delta \sigma_T^i(z)}{\lambda_i}, \quad i = 2 \dots N. \end{aligned} \quad (60)$$

In addition the corresponding Lamé part of the partition is such that

$$\begin{aligned} \left(\frac{r_i}{r_{i-1}} + \frac{1-\nu_T^i}{1+\nu_T^i} \frac{r_{i-1}}{r_i} \right) \frac{u_i}{r_i} - \frac{r_{i-1}}{r_i} \frac{2}{1+\nu_T^i} \frac{u_{i-1}}{r_{i-1}} = \\ \frac{1}{\lambda_i} \left(\frac{1-\nu_T^i}{E_T^i} \sigma_T^i - \frac{\nu_A^i}{E_A^i} \sigma_i + \alpha_T^i \Delta T \right), \quad i = 2 \dots N. \end{aligned} \quad (61)$$

Also, since $u_z = \varepsilon z$ for the Lamé solution it follows from (28) and (48) that $H_i(z) \equiv \varepsilon z$, $i = 1 \dots N$, for this special case.

From (53) and (56) it follows that

$$\frac{\Delta u_1(z)}{r_1} - \frac{1-\nu_T^1}{E_T^1} \Delta \sigma_T^1(z) - \frac{2\nu_A^1}{E_A^1} \frac{C_1(z)}{r_1} - \frac{1-\nu_T^1}{4E_T^1} r_1 C_1''(z) = 0, \quad (62)$$

and that for the corresponding Lamé part of the partition

$$\frac{u_1}{r_1} - \frac{1-\nu_T^1}{E_T^1} \sigma_T^1 + \frac{\nu_A^1}{E_A^1} \sigma_1 - \alpha_T^1 \Delta T = 0. \quad (63)$$

From (40) and (56), since $H_i'(z) = \varepsilon$, $i = 2 \dots N$, for the Lamé solution, it can be shown that

$$\begin{aligned}
\varepsilon_i(z) \equiv \varepsilon - H_i'(z) = \lambda_i \left(\frac{1}{\mu_A^i} - \frac{2\nu_A^i}{E_A^i} \right) \left[\bar{F}_2^i C_{i-1}''(z) - \bar{F}_1^i C_i''(z) \right] - \frac{2\lambda_i}{E_A^i} \left(\frac{C_{i-1}(z)}{r_i} - \frac{C_i(z)}{r_{i-1}} \right) \\
- \lambda_i \left(\bar{F}_2^i \Delta u_{i-1}''(z) - \bar{F}_1^i \Delta u_i''(z) \right) + \frac{4\lambda_i \mu_T^i \nu_A^i}{r_i^2 E_A^i} \left(r_i \Delta u_{i-1}(z) - r_{i-1} \Delta u_i(z) \right) \\
+ \frac{2\nu_A^i}{E_A^i} \Delta \sigma_T^i(z), \quad i = 2 \dots N,
\end{aligned} \quad (64)$$

and that for the corresponding Lamé part of the partition

$$\sigma_i = 2\nu_A^i \sigma_T^i + E_A^i (\varepsilon - \alpha_A^i \Delta T) + \frac{4\lambda_i \mu_T^i \nu_A^i}{r_i^2} (r_i u_{i-1} - r_{i-1} u_i), \quad i = 2 \dots N. \quad (65)$$

From (55) and (56) since $H_1'(z) = \varepsilon$ for the Lamé solution it follows that

$$\begin{aligned}
\varepsilon_1(z) \equiv \varepsilon - H_1'(z) = - \left(\frac{1}{\mu_A^1} - \frac{2\nu_A^1}{E_A^1} \right) \frac{r_1}{4} C_1''(z) + \frac{2}{r_1 E_A^1} C_1(z) \\
+ \frac{r_1}{4} \Delta u_1''(z) + \frac{2\nu_A^1}{E_A^1} \Delta \sigma_T^1(z),
\end{aligned} \quad (66)$$

and that for the corresponding Lamé part of the partition

$$\sigma_1 = 2\nu_A^1 \sigma_T^1 + E_A^1 (\varepsilon - \alpha_A^1 \Delta T). \quad (67)$$

Since u_i , $i = 1 \dots N$ for the Lamé solution are independent of z the relation (29), ensuring the continuity of u_z on $r = r_i$, $i = 1 \dots N-1$, may be written as the following recurrence relation defining $H_i(z)$, $i = 2 \dots N$, in terms of $H_1(z)$ and other functions

$$H_i(z) = H_{i-1}(z) + \alpha_i \left(\frac{C_i'(z)}{\mu_A^i} - \Delta u_i'(z) \right) - \beta_i \left(\frac{C_{i-1}'(z)}{\mu_A^i} - \Delta u_{i-1}'(z) \right), \quad (68)$$

where use has been made of (58).

10 GENERATING THE LAMÉ SOLUTION

The Lamé solution is defined by the relations (59), (61), (63), (65) and (67), together with the boundary conditions (10) or (11). On substituting (65) in (61) it follows, using (14), that the Lamé solution is generated from the following recurrence relation which must be iterated alternately with (59) for values $i = N \dots 2$

$$u_{i-1} = \frac{k_T^i r_i^2 + \mu_T^i r_{i-1}^2}{k_T^i + \mu_T^i} \frac{u_i}{r_i r_{i-1}} + \frac{k_T^i (r_i^2 - r_{i-1}^2)}{(k_T^i + \mu_T^i) r_{i-1}} \left(v_A^i \varepsilon - (\alpha_T^i + v_A^i \alpha_A^i) \Delta T - \frac{\sigma_T^i}{2 k_T^i} \right), \quad (69)$$

where k_T^i is the transverse bulk modulus of the i^{th} cylindrical element defined by

$$\frac{1}{2k_T^i} = \frac{1 - \nu_T^i}{E_T^i} - \frac{2(\nu_A^i)^2}{E_A^i}, \quad i = 1 \dots N. \quad (70)$$

It follows on substituting (67) in (63) that

$$\frac{u_1}{r_1} + v_A^1 \varepsilon - (\alpha_T^1 + v_A^1 \alpha_A^1) \Delta T - \frac{\sigma_T^1}{2k_T^1} = 0. \quad (71)$$

It is clear that if the values of σ_T^N and u_N are known, then the recurrence relations (59) and (69), when iterated alternately, generate the values of σ_T^i and u_i , $i = N-1 \dots 1$. The problem that remains is to ensure that the relation (71) is satisfied. Because of the linear nature of these recurrence relations it is clear that (71) will be of the form

$$f(\sigma_T^N, u_N) = \alpha + \beta \sigma_T^N + \gamma u_N = 0. \quad (72)$$

The values of α , β and γ are readily obtained numerically by making use of the following relations

$$\alpha = f(0, 0), \quad \beta = f(1, 0) - f(0, 0), \quad \gamma = f(0, 1) - f(0, 0), \quad (73)$$

where the function f is defined using the L.H.S. of (71).

From (10), (11) and (56) it is clear that the values of σ_T^N and u_N specified by the boundary conditions depend on the type of problem that is being solved. The values determined by the boundary conditions (10) and (11) are as follows:

$$\text{PROBLEM I:} \quad \sigma_T^N = \sigma_T, \quad (74)$$

$$\text{PROBLEM II:} \quad u_N = u. \quad (75)$$

It then follows from (72) that the values of σ_T^N and u_N corresponding to the values specified by (74) and (75) are given by

$$\text{PROBLEM I: } u_N = - \frac{\alpha + \beta \sigma_T}{\gamma}, \quad (76)$$

$$\text{PROBLEM II: } \sigma_T^N = - \frac{\alpha + \gamma u}{\beta}. \quad (77)$$

Thus by making use of the initial values specified by (74-77) the numerical values of σ_T^i and u_i , $i = N-1 \dots 1$ are readily obtained for problems I and II by applying the recurrence relations (59) and (69) alternately, starting with (69) as u_{i-1} is required by (59).

The corresponding values of σ_i , $i = 1 \dots N$ are then obtained from (65) and (67). The effective (or average) stress σ applied in the z -direction to the assembly of cylinders is given by

$$\sigma = \frac{r_1^2}{r_N^2} \sigma_1 + \sum_{i=2}^N \frac{r_i^2 - r_{i-1}^2}{r_N^2} \sigma_i. \quad (78)$$

11 GENERATING THE PERTURBATION SOLUTION

The result (60) is now written in the form of a recurrence relation generating values of the perturbations $\Delta u_i(z)$, $i = N-1 \dots 1$ as follows

$$\begin{aligned} \frac{\Delta u_{i-1}(z)}{r_{i-1}} = & \frac{1}{2} \left(\frac{r_i^2}{r_{i-1}^2} (1+v_T^i) + (1-v_T^i) \right) \frac{\Delta u_i(z)}{r_i} - \frac{1-(v_T^i)^2}{2E_T^i} \frac{r_i}{\lambda_i r_{i-1}} \Delta \sigma_T^i(z) \\ & + \frac{v_A^i (1+v_T^i)}{E_A^i} \frac{r_i}{r_{i-1}} \left(\frac{C_{i-1}(z)}{r_i} - \frac{C_i(z)}{r_{i-1}} \right) \\ & - \frac{1-(v_T^i)^2}{2E_T^i} \frac{r_i}{r_{i-1}} \left(\bar{F}_1^i C_i''(z) - \bar{F}_2^i C_{i-1}''(z) \right), \quad i = N \dots 2. \end{aligned} \quad (79)$$

Following a similar approach to that in the previous section, if $\Delta \sigma_T^N(z)$ and $\Delta u_N(z)$ are known, then the recurrence relations (57) and (79), when iterated alternately, generate the values of $\Delta \sigma_T^i(z)$ and $\Delta u_i(z)$, $i = N-1 \dots 1$. The problem that remains is to ensure that the relation (62) is satisfied. Because of the linear nature of these recurrence relations it is clear that (62) will be of the form

$$g(z, \Delta \sigma_T^N(z), \Delta u_N(z)) = \alpha(z) + \beta(z) \Delta \sigma_T^N(z) + \gamma(z) \Delta u_N(z) = 0, \quad (80)$$

where the functions $\alpha(z)$, $\beta(z)$ and $\gamma(z)$ are extremely complicated functions of z . The values of these functions are, however, readily obtained numerically by making use of the following relations

$$\begin{aligned} \alpha(z) &= g(z, 0, 0), \\ \beta(z) &= g(z, 1, 0) - g(z, 0, 0), \\ \gamma(z) &= g(z, 0, 1) - g(z, 0, 0), \end{aligned} \quad (81)$$

where the function g is defined by the L.H.S. of (62).

From (10), (11) and (56) it is clear that the values of $\Delta\sigma_T^N(z)$ and $\Delta u_N(z)$ specified by the boundary conditions depend on which problem is being solved. The values determined by (56) and the boundary conditions (10) and (11) are as follows:

$$\text{PROBLEM I: } \Delta\sigma_T^N(z) \equiv 0, \quad (82)$$

$$\text{PROBLEM II: } \Delta u_N(z) \equiv 0. \quad (83)$$

It then follows from (80) that the values of $\Delta\sigma_T^N(z)$ and $\Delta u_N(z)$ corresponding to the specified values given by (82) and (83) are given by

$$\text{PROBLEM I: } \Delta u_N(z) = -\frac{\alpha(z)}{\gamma(z)}, \quad (84)$$

$$\text{PROBLEM II: } \Delta\sigma_T^N(z) = -\frac{\alpha(z)}{\beta(z)}. \quad (85)$$

Thus by making use of the initial values specified by (82-85) the numerical values of $\Delta\sigma_T^i(z)$ and $\Delta u_i(z)$, $i = N-1 \dots 1$ are readily obtained by iterating the recurrence relations (57) and (79) alternately, starting with (79) since Δu_{i-1} is required by (57).

12 DERIVATION OF SIMULTANEOUS DIFFERENTIAL EQUATIONS

The various recurrence relations and initial values derived above, can be used to generate the functions $\Delta\sigma_T^i$, Δu_i and ε_i , $i = 1 \dots N$, in terms of the quantities C_i'''' , C_i'' and C_i , $i = 1 \dots N$. The fourth derivatives of the functions C_i arise because (79) must be differentiated twice with respect to z and then substituted into (64) and (66). From (10) and (11), since σ_{rz} is zero on the outer surface of the assembly of cylinders, it follows from (15) that, $C_N'(z) \equiv 0$. As the function $C_N(z)$ characterises the perturbation of the stress and displacement fields from the Lamé solution, it then follows that $C_N(z) \equiv 0$. Thus, the remaining unknowns of the problem are now $C_i(z)$, $i = 1 \dots N-1$.

It follows on subtracting $H_1(z)$ from both sides of (68), and then differentiating with respect to z , using the definitions of $\varepsilon_i(z)$ given by (64) and (66), that

$$\Delta\varepsilon_i(z) = \Delta\varepsilon_{i-1}(z) - \alpha_i \left(\frac{C_i''(z)}{\mu_A^i} - \Delta u_i''(z) \right) + \beta_i \left(\frac{C_{i-1}''(z)}{\mu_A^i} - \Delta u_{i-1}''(z) \right), \quad i = 2 \dots N, \quad (86)$$

where

$$\Delta\varepsilon_i(z) \equiv \varepsilon_i(z) - \varepsilon_1(z), \quad i = 2 \dots N. \quad (87)$$

The relation (86) is a recurrence relation which, when iterated using the initial value $\Delta\varepsilon_1(z) \equiv 0$, generates the functions $\Delta\varepsilon_i(z)$, $i = 2 \dots N$.

On substituting the values of $\varepsilon_i(z)$, $i = 1 \dots N$ given by (64) and (66) into the R.H.S. of (87), and on substituting the values of $\Delta\varepsilon_i(z)$ generated by (86) on the L.H.S. of (87), the following fourth order ordinary differential equations are derived

$$\sum_{i=1}^{N-1} F_{ij} C_i''''(z) + \sum_{i=1}^{N-1} G_{ij} C_i''(z) + \sum_{i=1}^{N-1} H_{ij} C_i(z) \equiv 0, \quad j = 1 \dots N-1, \quad (88)$$

where the coefficients F_{ij} , G_{ij} and H_{ij} , $i, j = 1 \dots N-1$, are very complicated expressions generated by recurrence relations. Their values can, however, be calculated using numerical methods as described in the following section. It should be noted that (88) has been derived from the derivative of (68). Thus, solutions of (88) will lead to values of u_z such that $\partial u_z / \partial z$ is automatically continuous at the interfaces $r = r_i$, $i = 1 \dots N-1$.

13 CALCULATION OF THE COEFFICIENTS

It is convenient to denote the L.H.S. of the j^{th} equation of the set (88) by $P_j(C, C'', C''')$, where $C \equiv \{C_i, i = 1 \dots N-1\}$, having second and fourth derivatives denoted by C'' and C'''' respectively. The numerical values of the coefficients F_{ij} , G_{ij} and H_{ij} , $i, j = 1 \dots N-1$, are then calculated simply by obtaining values of $P_j(z)$, $j = 1 \dots N-1$ using suitable sets of values for the functions C , C'' and C'''' . Thus:

$$\begin{aligned} F_{ij} &\equiv P_j(0, 0, I_i), \\ G_{ij} &\equiv P_j(0, I_i, 0), \quad i = 1 \dots N-1, \quad j = 1 \dots N-1, \\ H_{ij} &\equiv P_j(I_i, 0, 0), \end{aligned} \quad (89)$$

with $I_i \equiv \{\delta_{ik}, k = 1 \dots N-1\}$ where δ_{ik} is the Kronecker delta, and with 0 denoting a corresponding set of zero values. Using values of the coefficients resulting from such calculations the functions C_i , $i = 1 \dots N-1$, can then be determined from (88) using numerical methods [4] as soon as appropriate additional boundary conditions are imposed.

14 AXIAL BOUNDARY CONDITIONS INVOLVING ONLY STRESSES

For many axial stress transfer problems it is possible to impose stress-free or symmetry boundary conditions on two planes so that

$$\sigma_{rz}(r, \pm L) = 0, \quad \text{for all } r, \quad (90)$$

where $2L$ is the separation of the planes. Since $C_N(z) \equiv 0$ it then follows from (15) that

$$C_i'(\pm L) = 0, \quad i = 1 \dots N-1. \quad (91)$$

On the planes $z = \pm L$ the following conditions are applied to the axial stress component

$$\sigma_{zz} = \begin{cases} s_1^\pm, & \text{for } 0 \leq r \leq r_1, \\ s_i^\pm, & \text{for } r_{i-1} \leq r \leq r_i, \quad i = 2 \dots N. \end{cases} \quad (92)$$

The values of $s_i^\pm$, $i = 1 \dots N$, must be such that the system will be in mechanical equilibrium, i.e. the following relations must be satisfied, since from (10) and (11) $\sigma_{rz} = 0$ on $r = r_N$ for all values of z ,

$$r_1^2 s_1^+ + \sum_{i=2}^N (r_i^2 - r_{i-1}^2) s_i^+ = r_1^2 s_1^- + \sum_{i=2}^N (r_i^2 - r_{i-1}^2) s_i^- = r_N^2 \sigma, \quad (93)$$

where σ is the effective stress applied to the composite, defined by (78).

On substituting the values specified by (92) into the relations (25) and (45) it can be shown that the following recurrence relation defines the remaining boundary conditions that must be imposed so that the simultaneous ordinary differential equations (88) have a unique solution,

$$C_i(\pm L) = \frac{r_{i-1}}{r_i} C_{i-1}(\pm L) + \frac{r_{i-1}}{2\lambda_i} (\sigma_i - s_i^{\pm}), \quad i = 2 \dots N-1, \quad (94)$$

where

$$C_1(\pm L) = \frac{1}{2} r_1 (\sigma_1 - s_1^{\pm}). \quad (95)$$

It is clear from (94) and (95) that if $s_i^{\pm} = \sigma_i$, $i = 1 \dots N$, then $C_i(\pm L) = 0$, $i = 1 \dots N-1$. Thus, it follows on using (91) that the solution of the simultaneous differential equations (88) must be given by $C_i(z) \equiv 0$, $i = 1 \dots N-1$, i.e. there is no stress transfer. Other values of the parameters $s_i^{\pm}$, $i = 1 \dots N$, satisfying (93) will lead to stress transfer effects.

15 MIXED AXIAL BOUNDARY CONDITIONS

For many stress transfer problems involving multiple concentric cylinders the axial boundary conditions will involve displacements as well as stresses. As an example a problem will now be considered where the stress and displacement distribution is sought in the region between two adjacent fractures in a fibre belonging to a unidirectional composite.

The fibre is represented by k cylinders occupying the regions

$$r_{i-1} \leq r \leq r_i, \quad i = 1 \dots k, \quad \text{where } r_0 = 0.$$

The matrix and surrounding composite material is represented by $N - k$ cylinders occupying the regions

$$r_{i-1} \leq r \leq r_i, \quad i = k+1 \dots N.$$

If K cylinders are chosen to represent the matrix then clearly $N - k - K$ cylinders represent the surrounding composite material.

Assume that the fibre is cracked into fragments each of length $2L$, and consider only one of these fragments and the associated matrix and surrounding composite. Because the fibre fracture surface is stress-free and also because of symmetry it follows from (15) that $C_N'(z) \equiv 0$ and that

$$C_i'(\pm L) = 0, \quad i = 1 \dots N-1. \quad (96)$$

Since $\sigma_{zz} \equiv 0$ on $z = \pm L$ for $0 \leq r \leq r_k$ for a broken fibre it follows from (25) and (45) that

$$C_i(\pm L) = \frac{r_{i-1}}{r_i} C_{i-1}(\pm L) + \frac{r_{i-1}}{2\lambda_i} \sigma_i, \quad i = 2 \dots k, \quad (97)$$

where

$$C_1(\pm L) = \frac{1}{2} r_1 \sigma_1, \quad (98)$$

thus specifying the values of $C_i(\pm L)$, $i = 1 \dots k$.

On the planes $z = \pm L$ the displacement component u_z should be uniform in the region $r_k \leq r \leq r_N$. It is not possible to satisfy this condition exactly if it is demanded that $\sigma_{rz} = 0$ on these planes (see section 16 for further discussion of this aspect of the problem). However, for the cylinders representing the matrix and surrounding composite, average displacement conditions can be satisfied and take the form

$$\bar{u}_z^i(\pm L) = \bar{u}_z^{i-1}(\pm L), \quad i = k+2 \dots N. \quad (99)$$

The total number of conditions specified is $4(N-1)$ as required by the differential equations (88) in order to have a unique solution.

The average displacement conditions can also be written

$$\Delta \bar{u}_z^i(\pm L) \equiv \Delta \bar{u}_z^{i-1}(\pm L), \quad i = k+2 \dots N, \quad (100)$$

where

$$\Delta \bar{u}_z^i(z) = \bar{u}_z^i(z) - \varepsilon z, \quad i = 1 \dots N. \quad (101)$$

On setting

$$\Delta H_i(z) \equiv H_i(z) - \varepsilon z, \quad i = 1 \dots N, \quad (102)$$

where in fact from (64) and (66)

$$\Delta H_i'(z) \equiv -\varepsilon_i(z), \quad i = 1 \dots N, \quad (103)$$

it can be shown using (39) that for $r_{i-1} \leq r \leq r_i$, $i = 2 \dots N$

$$\begin{aligned} \Delta \bar{u}_z^i(z) = & \lambda_i \left[\frac{C_{i-1}'(z)}{\mu_A^i} - \Delta u_{i-1}'(z) \right] \bar{F}_2^i \\ & - \lambda_i \left[\frac{C_i'(z)}{\mu_A^i} - \Delta u_i'(z) \right] \bar{F}_1^i + \Delta H_i(z), \end{aligned} \quad (104)$$

and on using (54) that for $0 \leq r \leq r_1$

$$\Delta \bar{u}_z^1(z) = -\frac{r_1}{4} \left[\frac{C_1'(z)}{\mu_A^1} - \Delta u_1'(z) \right] + \Delta H_1(z). \quad (105)$$

It can then be shown for $r_{j-1} \leq r \leq r_j$, $j = 1 \dots N$ (with $r_0 = 0$) that in general

$$\Delta \bar{u}_z^j(z) = \sum_{i=1}^N f_{ij} C_i'''(z) + \sum_{i=1}^N g_{ij} C_i'(z) + \sum_{i=1}^N h_{ij} \bar{C}_i(z), \quad (106)$$

where

$$\bar{C}_i(z) \equiv \int_0^z C_i(z') dz', \quad i = 1 \dots N. \quad (107)$$

It should be noted from (27) and (47) that the displacement component u_z depends upon the functions $H_i(z)$, $i = 1 \dots N$. These functions can be obtained from (64) and (66) on integrating with respect to z , thus introducing the functions $\bar{C}_i(z)$, $i = 1 \dots N$. By defining the functions $\bar{C}_i$ such that $\bar{C}_i(0) = 0$ for all $i = 1 \dots N-1$, and on observing that symmetry will lead to $C_i'(0) = C_i'''(0) = 0$, $i = 1 \dots N$, the displacement component u_z will have a zero value on the plane $z = 0$ for fibre, matrix and surrounding composite. In addition u_z will be automatically continuous at the interfaces $r = r_i$, $i = 1 \dots N-1$, thus satisfying the boundary condition (9)₂. For the boundary conditions specified by (10) and (11) the function $\bar{C}_N(z) \equiv 0$.

The functions $\Delta H_i(z)$ can be calculated by two equivalent methods. First of all use can be made of (68) and (102) to set up the following recurrence relation

$$\Delta H_i(z) = \Delta H_{i-1}(z) + \alpha_i \left(\frac{C_i'(z)}{\mu_A^i} - \Delta u_i'(z) \right) - \beta_i \left(\frac{C_{i-1}'(z)}{\mu_A^i} - \Delta u_{i-1}'(z) \right), \quad i = 2 \dots N, \quad (108)$$

where from the integral of (103) with $i = 1$

$$\Delta H_1(z) \equiv -\bar{\epsilon}_1(z) = -\int_0^z \epsilon_1(z') dz'. \quad (109)$$

For this method of calculation the resulting expressions for $\Delta \bar{u}_z^j(z)$ are of the form specified by (106).

For the second method of calculation the functions $\Delta H_i(z)$ are derived by integrating (103) with respect to z for all i so that

$$\Delta H_i(z) \equiv -\bar{\epsilon}_i(z) = -\int_0^z \epsilon_i(z') dz' \quad i = 1 \dots N, \quad (110)$$

since $C_i'(0) = C_i'''(0) = \bar{C}_i(0) = 0$. For this case the resulting expressions for $\Delta \bar{u}_z^j(z)$ are of the simpler form specified by

$$\Delta \bar{u}_z^{(j)}(z) = \sum_{i=1}^N g_{ij} C_i'(z) + \sum_{i=1}^N h_{ij} \bar{C}_i(z), \quad (111)$$

where now the coefficients f_{ij} are all zero. The third derivatives of the functions $C_i(z)$ are not present because of the following result which can be obtained by integrating the differential equations (88)

$$\sum_{i=1}^{N-1} F_{ij} C_i'''(z) + \sum_{i=1}^{N-1} G_{ij} C_i'(z) + \sum_{i=1}^{N-1} H_{ij} \bar{C}_i(z) \equiv 0, \quad j = 1 \dots N-1. \quad (112)$$

The coefficients f_{ij} , g_{ij} and h_{ij} appearing in (106) and (111) can be calculated numerically using techniques that are very similar to those used when deriving the coefficients of the differential equations (88).

16 A SPECIAL CASE

For the problem of stress transfer associated with the fracture of a fibre that is surrounded by a region of matrix and then a region of effective composite material, there is an alternative approach that leads to more accurate solutions. The approach for this special case will now be described. In section 15, the boundary conditions (90) were satisfied for all cylinders of the representation. The new approach is to select the cylinders on either side of the fibre/matrix interface to be very thin compared to the fibre radius. The conditions (97) and (98), are retained while the boundary conditions (90) are replaced by

$$\sigma_{rz}(r, \pm L) = 0, \quad \text{for } 0 \leq r \leq r_{k-1}, \quad r_{k+1} \leq r \leq r_N, \quad (113)$$

in which case, on remembering that $C_N(z) \equiv 0$, (96) is replaced by

$$C_i'(\pm L) = 0, \quad i = 1 \dots k-1, k+1 \dots N-1. \quad (114)$$

Clearly $C_k'(\pm L) \neq 0$ so that $\sigma_{rz} \neq 0$ for $r_{k-1} \leq r \leq r_{k+1}$. This means that the boundary condition (90) is not satisfied in a small zone in the immediate neighbourhood of the fibre/matrix interface provided that $r_{k-1} \neq r_k$ and $r_{k+1} \neq r_k$.

In this new approach an attempt is made to satisfy the following displacement boundary condition exactly

$$u_z(r, \pm L) = \pm \text{constant}, \quad \text{for } r_k \leq r \leq r_N. \quad (115)$$

This condition asserts that the magnitude of the normal displacement of the matrix and the surrounding composite in the planes of fibre fractures must have a single value that is found as part of the solution to the problem. On differentiating (115) with respect to r the following equivalent boundary condition is obtained

$$\frac{\partial u_z}{\partial r} = 0 \quad \text{on } z = \pm L \quad \text{for } r_k \leq r \leq r_N. \quad (116)$$

From (26) and (56) it can be shown that (116) is satisfied if the following two conditions are met

$$\Delta u'_{i-1}(\pm L) = \frac{C'_{i-1}(\pm L)}{\mu_A^i}, \quad i = k+1 \dots N, \quad (117)$$

$$\Delta u'_i(\pm L) = \frac{C'_i(\pm L)}{\mu_A^i}, \quad i = k+1 \dots N, \quad (118)$$

where it should be remembered that $C_N(z) \equiv 0$ in which case from (118) $\Delta u'_N(z) \equiv 0$ consistent with the initial value (83) for the recurrence relations (79), i.e. the special case corresponds to Problem II described in sections 10 and 11.

Thus, for the special case under discussion, the boundary conditions that have been applied demand that $u(r_N, z) = u$ is specified to have a fixed value corresponding to that which would result when the assembly of cylinders does not contain any cracks but is subject to the same applied tractions and has the same temperature as the assembly containing cracks. It follows from (114), (117) and (118), and the fact that $\Delta u_N(z) \equiv 0$ for Problem II, that the required conditions have the following form

$$\Delta u'_k(\pm L) = \frac{C'_k(\pm L)}{\mu_A^{k+1}}, \quad (119)$$

$$\Delta u'_i(\pm L) = 0, \quad i = k+1 \dots N-1. \quad (120)$$

It can be shown from the differentials of (57) and (79) with respect to z that the functions $\Delta u'_j$ are of the form

$$\Delta u'_j(z) \equiv \sum_{i=1}^{N-1} D_{ij} C_i'''(z) + \sum_{i=1}^{N-1} E_{ij} C'_i(z), \quad j=1 \dots N-1, \quad (121)$$

where the coefficients D_{ij} and E_{ij} are readily calculated numerically by iteration with suitable sets of initial values using an analogous procedure to that described in section 13.

It can be seen from (97), (98), (114), (119) and (120) that $4N - 4$ conditions have been imposed, corresponding to the number of conditions needed to ensure that the system (88) of fourth order ordinary differential equations has a unique solution.

17 STRESS TRANSFER ASSOCIATED WITH FIBRE FRACTURE

An axisymmetric representative volume element of a unidirectional composite is now considered where a fibre of radius $R = 3.5 \mu\text{m}$ is surrounded by matrix material such that the volume fraction is given by $V_f = 0.6$, i.e. the outer radius of matrix material has the value $a = R/\sqrt{V_f} = 4.51848 \mu\text{m}$. The thermoelastic constants assumed for the fibre and matrix are given in Table 1. The fibre and matrix are embedded in an anisotropic medium that has the thermoelastic constants of the undamaged composite. The interfaces between fibre and matrix, and between matrix and the surrounding effective composite material, are assumed to be perfectly bonded. The thermoelastic constants of the undamaged composite have been calculated using methods described in an earlier report [1], and their values are also given in Table 1.

In order to obtain an accurate solution of the stress transfer problem the fibre, matrix and surrounding effective composite are represented by series of concentric cylinders perfectly bonded together. The fibre is represented by 8 cylinders, the matrix by 4 cylinders and the surrounding composite by 9 cylinders where the outer radii are given in Table 2. It is assumed that there are fibre fractures on the planes $z = \pm L$ so that the problem solved corresponds to a fibre that is fragmented into sections of equal length $2L$. The value of L selected is $350 \mu\text{m}$ i.e. 100 fibre radii. This selection was made so that the fibre fractures are far enough apart to avoid any interactions. Clearly the model could easily be used to investigate interaction effects that are known to occur for smaller values of L that arise when fibre fragmentation tests are conducted. The stress-free temperature of the composite is taken to be 150°C so that the parameter $\Delta T = -130^\circ\text{C}$, assuming that room temperature is 20°C . The composite is subject to a uniaxial load that induces a uniform axial strain percent of 1.49595% in an undamaged composite. The corresponding axial stress in the fibres is 3.505 GPa. This is identical to the FEA solution [3, 5] as $z \rightarrow 0$ because the applied strain percent for the analytical model of 1.49595% is chosen so that the fibre stress remote from the fibre fractures corresponds closely to the FEA value. The corresponding radial displacement of the outer boundary of the assembly of cylinders is given by $u_r(r_N, z) = -0.2780663 \mu\text{m}$. This is the value of u_r needed to calculate initial values for the recurrence relations (see (75) and (77)). This selection of the applied strain ensures that the solution corresponds to FEA calculations [3, 5] carried out where the strain applied to a damaged composite is 1.5%. Analytical solutions have been sought that satisfy the boundary conditions specified in section 16.

The more relevant results of the calculations are given in Fig.2a-g, where comparisons are made with the finite element solutions [3, 5]. Fig.2a shows the dependence of the value of the axial stress σ_{zz} at the centre of the broken fibre as a function of the distance measured along the fibre. The fibre fracture is located at $z = 350 \mu\text{m}$ and it is seen that the axial stress is zero as required by the boundary conditions. The point $z = 0$ corresponds to the mid-point between two fibre breaks located at $z = \pm 350 \mu\text{m}$. In the region $0 \leq z \leq 100 \mu\text{m}$ the axial stress is virtually independent of z indicating that the solution obtained applies also to a single isolated fibre fracture. The results shown in Fig.2a indicate that there is very good agreement between the predictions made using analytical and FEA methods of solution. A remarkable feature of the results is that stress transfer between fibre and matrix is occurring over a distance of about $150 \mu\text{m}$ which corresponds to over 40 fibre radii. The large stress transfer distance results from the inclusion of the effect of surrounding effective composite material on the stress transfer that occurs in the neighbourhood of the fibre break. Corresponding solutions where the presence of the effective composite material is neglected show (see Fig.3) that stress transfer then occurs over a distance of about 8 fibre radii, i.e. stress transfer is much more localised. The solution shown in Fig.3 used 9 cylinders to represent the fibre and nine to represent the matrix. On the external surface $r = r_N$ the condition $\sigma_{rr} = 0$ was imposed corresponding to Problem I.

Fig.2b shows the dependence of the shear stress σ_{rz} on the fibre/matrix interface ($r = R$) as a function of distance along the interface. The shear stress has non-zero values only in regions where stress transfer between fibre and matrix is occurring. Again there is very good agreement between the results of analytical and FEA methods. However, very close to the fibre fracture both analytical and FEA predictions break down showing oscillatory behaviour. This phenomenon is occurring because neither the analytical nor the FEA techniques have taken into account the singularity that the exact solution must possess. The singularity enables the solution to predict that on the surface of the fibre fracture the shear stress $\sigma_{rz} = 0$ as $r \rightarrow R$, while on the fibre/matrix interface $r = R$ the shear stress σ_{rz} tends to a finite value as $z \rightarrow L$. It is clear from Fig.2b that the analytical solution is giving more reliable results near the fibre fracture than the FEA solution. Fig.2c shows the dependence of the shear stress σ_{rz} on the matrix/composite interface as a function of distance along the interface. For this prediction the analytical method is predicting a higher maximum shear stress than the FEA method.

Fig.2d shows the distribution of the normal stress σ_r on the fibre/matrix interface together with the finite element results which are shown as a dotted line. Fig. 2e shows the corresponding distribution for matrix/composite interface. The small discrepancy in Figs. 2d and 2e between analytical and FEA results as $z \rightarrow 0$ is thought to arise because the FEA analysis is based on thermoelastic constants for the effective composite material that differ slightly from those used for the analytical technique. The FEA values were chosen so that the thermoelastic constants for effective composite material matched exactly those predicted by the FEA analysis of an undamaged square array of fibres (see Table 3). In contrast, the analytical results are based on thermoelastic constants for the effective composite that have been found from a concentric cylinders model of a composite (see Table 2) where the fibre is represented by one cylinder and the matrix by a second cylinder that is concentric with the fibre. This exact matching leads to the result shown in Fig.2e where $\sigma_r \rightarrow 0$ as $z \rightarrow 0$.

Fig.2f shows the dependence of the axial stress σ_{zz} in the plane of the fibre fracture on the radial distance r measured from the centre of the fibre. The step function corresponds to the analytical predictions; the other curve being the FEA solution. In the region $0 \leq r \leq 3.5 \mu\text{m}$ the analytical solution predicts a zero value for the axial stress thus satisfying the required boundary condition exactly. The FEA solution in this region exhibits non-zero values that violate the boundary condition. In the matrix material at the fibre/matrix interface very high tensile stresses are encountered. The maximum value predicted by FEA is 15.9 GPa and that by the analytical technique is 16.07 GPa thus showing very good agreement. The exact solution would predict an infinite axial stress at this point resulting from the singularity in the stress field. It is seen that in the surrounding effective composite material the axial stress reduces from maximum value (4.050 GPa for analytical solution and 3.667 GPa for FEA) at the matrix/composite interface to a constant value of 2.1424 GPa (same value for analytical and FEA predictions). There is reasonably good agreement between analytical and FEA predictions, although the analytical results are thought to be the more reliable.

Fig.2g plots the axial displacement in the plane of the fibre fracture relative to the uniform axial displacement of matrix and effective composite material. Thus for the region $r \geq 3.5 \mu\text{m}$ both FEA and analytical methods predict a zero value, thus satisfying the required displacement boundary condition. For the region $0 \leq r \leq 3.5 \mu\text{m}$ the analytical method predicts a larger displacement difference than the FEA method. Of particular note is the very large displacement gradient that is observed in the region of the fibre/matrix interface, indicating again the effect of the singularity in the stress field.

18 ESTIMATION OF STRESS CONCENTRATION FACTORS

When an isolated fibre fracture occurs there are two methods by which the new stress transfer model can be used to estimate the increase in axial stress experienced by the fibres which are nearest neighbours. The first method is to use the solution described in section 17 to calculate the local axial stress distribution at the location in the effective composite corresponding to the axes of the nearest neighbour fibres. For a hexagonal arrangement of fibres, the distance $2d$ between the centres of the fibres is given by the formula

$$d = \sqrt{\frac{\pi}{2\sqrt{3}} \frac{R^2}{V_f}} \quad (122)$$

Thus for the problem formulated above $d = 4.303 \mu\text{m}$ and the axial stress distribution at the location of the centres of the nearest neighbour fibres is shown in Fig.4. The maximum value of axial stress on $r = 2d$, corresponding to the location of the axes of the nearest neighbour fibres, has the value 2.268 GPa (FEA result is 2.243 GPa which is in very good agreement with the prediction of the analytical model). The corresponding stress value remote from the plane containing the fibre break is 2.1424 GPa (FEA result is 2.149 GPa). Thus the stress

concentration factor in the nearest neighbour fibres has the value 1.059 when using the analytical model, and the value 1.044 when using FEA.

The second approach to estimating stress concentration factors associated with isolated fibre fractures is to smooth out the six nearest fibres in a hexagonal array into the form of a hollow cylinder or tube. The area of cross-section of the tube has the value $6\pi R^2$ and it occupies the region $2d - \delta \leq r \leq 2d + \delta$, where

$$\delta = \frac{3R^2}{4d}. \quad (123)$$

For the current problem the value of δ is $2.13514 \mu\text{m}$.

The tube representing the nearest neighbours is then embedded in matrix occupying the regions $R \leq r \leq 2d - \delta$ and $2d + \delta \leq r \leq R_m$, where the outer radius of matrix material R_m is selected using the following formula, that ensures that the volume fraction of fibre material in the region $0 \leq r \leq R_m$ has the value V_f

$$R_m = \sqrt{\frac{7R^2}{V_f}}. \quad (124)$$

Thus for the problem of interest $R_m = 11.9548 \mu\text{m}$. The assembly representing the broken fibre, the nearest neighbour fibres, and the associated matrix are then embedded in effective composite material having the properties given in Table 3, and occupying the region $R_m \leq r \leq 30 \mu\text{m}$.

Fig.5 shows the axial stress distribution in the plane of the fibre break. It is seen that the maximum predicted stress in the cylindrical tube representing nearest neighbour fibres has the value 4.1049 GPa while the value of the axial stress as $z \rightarrow 0$ is 3.5104 GPa . Thus the stress concentration factor in nearest neighbour fibres resulting from an isolated fibre break has the value 1.1694. This value is larger than that predicted by the first method, and is very close to the value $7/6 = 1.1667$ which is the stress concentration factor predicted by simple local load sharing where the load carried by the fibre that is going to break is shared equally amongst the six nearest neighbour fibres. The value predicted by the analytical approach is higher than the local load sharing value because there is a stress concentration within the cylindrical tube representing the nearest neighbour fibres. An effective stress, based on the total load carried by the six nearest neighbour fibres divided by their cross sectional area $6\pi R^2$, can be calculated to be 3.7461 GPa , and the corresponding stress concentration factor is 1.0671 which is slightly lower than the local load sharing value. It is interesting to note from Fig.5 that the axial stress in the surrounding effective composite material is almost uniform suggesting that its presence is unnecessary if the representative volume element for the composite includes the nearest neighbour fibres. This supports the conclusion that the load carried by a breaking fibre is transferred mainly to the six nearest neighbours in a hexagonally packed unidirectional composite, and the result that the stress concentration factor is close to the local load sharing value of 1.1667.

19 CONCLUSIONS

1. The problem of predicting the stress and displacement distributions in a perfectly bonded multiple cylinder stress transfer model has been reduced to manipulating sets of recurrence relations that generate a set of fourth order simultaneous ordinary differential equations. Numerical solutions of the problem are readily obtained using a desk-top computer and experience has shown that the new technique is robust.
2. The technique allows each cylinder to be a transverse isotropic solid and takes account of the residual stresses arising from thermal expansion mismatch.
3. For a unidirectional composite with fibre fractures, by representing the fibres by one set of cylinders, the matrix by another set, and surrounding effective composite material by a third set, accurate solutions for stress and displacement distributions can be found in the vicinity of a fibre fracture that agree closely with corresponding finite element solutions. The analytical predictions are thought to be more reliable than FEA solutions for the perfectly bonded case.
4. For the fibre fracture stress transfer problem singularities are encountered for the axial stress in the matrix in the plane of the fibre fracture. These singularities are modelled reasonably well by the new technique whereas corresponding FEA predictions are subject to error in such regions.
5. Stress transfer between fibre and matrix is affected significantly if the representative volume for the unidirectional composite is embedded in effective composite material. Stress transfer then occurs over a much larger distance along the fibre.
6. By smoothing out nearest neighbour fibres into the form of a cylindrical tube or shell, the multiple cylinder model can be used to estimate the stress concentrations that are induced in neighbouring fibres when an isolated fibre fracture occurs in a unidirectional composite. The value of the stress concentration is slightly higher than that predicted by local load sharing if account is taken of the axial stress gradient in the nearest neighbour fibres, and slightly smaller if the stress concentration is based on the average stress in these fibres. These results indicate that the load carried by a breaking fibre in a hexagonally packed unidirectional carbon fibre epoxy composite is transferred mainly to its nearest neighbours.

REFERENCES

1. L N McCartney, 'Analytical method of calculating the thermoelastic constants of a multi-phase unidirectional composite', NPL Report DMM(A)57, March 1992.
2. L N McCartney, 'New theoretical model of stress transfer between fibre and matrix in a uniaxially fibre reinforced composite', Proc. Roy. Soc. Lond. A425, (1989), 215-244.
3. M Nedele and M Wisnom, 'Stress concentrations around broken fibres in unidirectional carbon fibre reinforced epoxy', presented at conference on 'Interfacial phenomena in composite materials', Cambridge, September 1993. (papers of conference to appear in special issue of 'Composites').
4. S A Hannaby, 'The solution of ordinary differential equations arising from stress transfer mechanics', NPL Report DITC 229/93, October 1993.
5. M Nedele, Specific FEA calculations, Private communication 1993.

ACKNOWLEDGEMENTS

The analytical approach developed in this report, and numerical algorithms used to solve the differential equations (88), were carried out as part of the NPL Strategic Research Programme. The author wishes thank Mr Simon Hannaby and Prof. George Symm, of the Division of Information Technology and Computing at NPL, for respectively carrying out the numerical analysis using MATLAB and checking the mathematical analysis. It is acknowledged that the actual problems solved in the report arose from participation in PREDICT. This project (Prediction of Damage Initiation and Growth in Composite Materials) is a collaboration between British Aerospace (Sowerby Research Centre), Ciba Geigy Plastics, Rolls Royce, Tenax Fibres, National Physical Laboratory, University of Bristol and the University of Surrey. The project is led by British Aerospace and is funded under the LINK Structural Composites programme of the DTI's Research and Technology Initiative (IED Grant Ref. RA 6/25/01).

	Fibre	Matrix	Composite
E_A (GPa)	235.0	3.89	142.614164
E_T (GPa)	16.7	3.89	9.08208123
ν_A	0.1437	0.37	0.227431817
ν_T	0.35	0.37	0.464011673
μ_A (GPa)	26.0	1.41971	4.72446013
α_A ($\times 10^6/^{\circ}\text{C}$)	- 0.25	58.0	0.483855705
α_T ($\times 10^6/^{\circ}\text{C}$)	25.0	58.0	45.0177201

TABLE 1 Values assumed for the thermoelastic constants for the fibre, matrix and surrounding composite representing a unidirectional carbon fibre reinforced epoxy composite having a volume fraction $V_f = 0.6$.

Outer radii of fibre elements (μm)	Outer radii of matrix elements (μm)	Outer radii of composite element (μm)
0.5	3.525	4.75
1.0	3.8	5.3
1.5	4.2	6.0
2.0	4.51848	7.0
2.5		8.0
3.0		9.0
3.475		10.0
3.5		20.0
		30.0

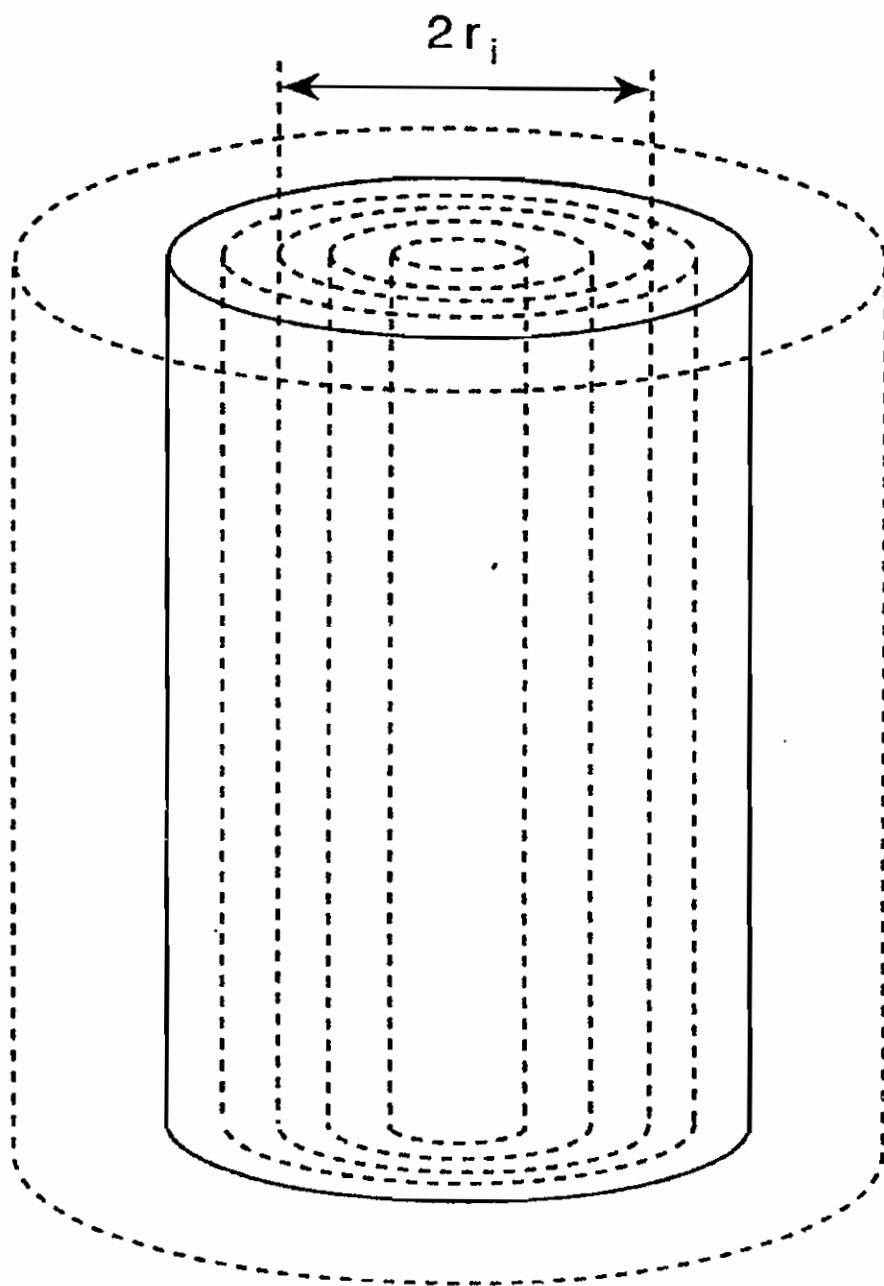
TABLE 2 Values assumed for the outer radii of the cylindrical elements representing fibre, matrix and surrounding composite having a volume fraction $V_f = 0.6$.

	4-cylinder model	FEA values
E_A (GPa)	142.656406	142.611
E_T (GPa)	9.18024364	9.818
ν_A	0.222501862	0.2260
ν_T	0.469069883	0.423
μ_A (GPa)	6.89111049	4.975
α_A ($\times 10^6/^{\circ}\text{C}$)	0.555019604	0.646
α_T ($\times 10^6/^{\circ}\text{C}$)	43.8170764	44.8

TABLE 3 Values for the thermoelastic constants, for the effective composite material representing a unidirectional carbon fibre reinforced epoxy composite having a volume fraction $V_f = 0.6$, used when comparing analytical and FEA predictions.

FIGURE CAPTIONS

- Fig.1 Schematic diagram of an assembly of concentric cylinders that are perfectly bonded together. The outer cylinder can represent effective composite material when analysing stress transfer caused by fibre fracture in a unidirectional composite.
- Fig.2a Distribution of the axial stress σ_{zz} along the axis of a fibre in a unidirectional composite that has an isolated fibre break at $z = 350 \mu\text{m}$. The corresponding FEA prediction is shown as a dotted line.
- Fig.2b Distribution of the fibre/matrix interfacial shear stress σ_{rz} in a unidirectional composite that has an isolated fibre break at $z = 350 \mu\text{m}$. The corresponding FEA prediction is shown as a dotted line.
- Fig.2c Distribution of the matrix/composite interfacial shear stress σ_{rz} in a unidirectional composite that has an isolated fibre break at $z = 350 \mu\text{m}$. The corresponding FEA prediction is shown as a dotted line.
- Fig.2d Distribution of the normal stress σ_{rr} on the fibre/matrix interface in a unidirectional composite that has an isolated fibre break at $z = 350 \mu\text{m}$. The corresponding FEA prediction is shown as a dotted line.
- Fig.2e Distribution of the normal stress σ_{rr} on the matrix/composite interface in a unidirectional composite that has an isolated fibre break at $z = 350 \mu\text{m}$. The corresponding FEA prediction is shown as a dotted line.
- Fig.2f Distribution (shown as a step function) of the axial stress σ_{zz} as a function of r in the plane of an isolated fibre break in a unidirectional composite. The corresponding FEA prediction is shown as a continuous line.
- Fig.2g Distribution of the difference in axial displacement, in the plane of an isolated fibre break, as a function of r for a unidirectional composite. The corresponding FEA prediction is shown as a dotted line.
- Fig.3 Distribution of the axial stress σ_{zz} along the axis of a fibre in a unidirectional composite that has fibre breaks at $z = \pm 350 \mu\text{m}$ for the case when account is not taken of surrounding effective composite material.
- Fig.4 Distribution of the axial stress σ_{zz} in the effective composite material at the location of the axes of the nearest neighbour fibres in a unidirectional composite that has an isolated fibre break at $z = 350 \mu\text{m}$.
- Fig.5 Distribution (shown as a step function) of the axial stress σ_{zz} in the plane of an isolated fibre break for a unidirectional composite where the nearest neighbour fibres have been smoothed out into the form of a cylindrical tube that is concentric to the broken fibre.



. Fig.1

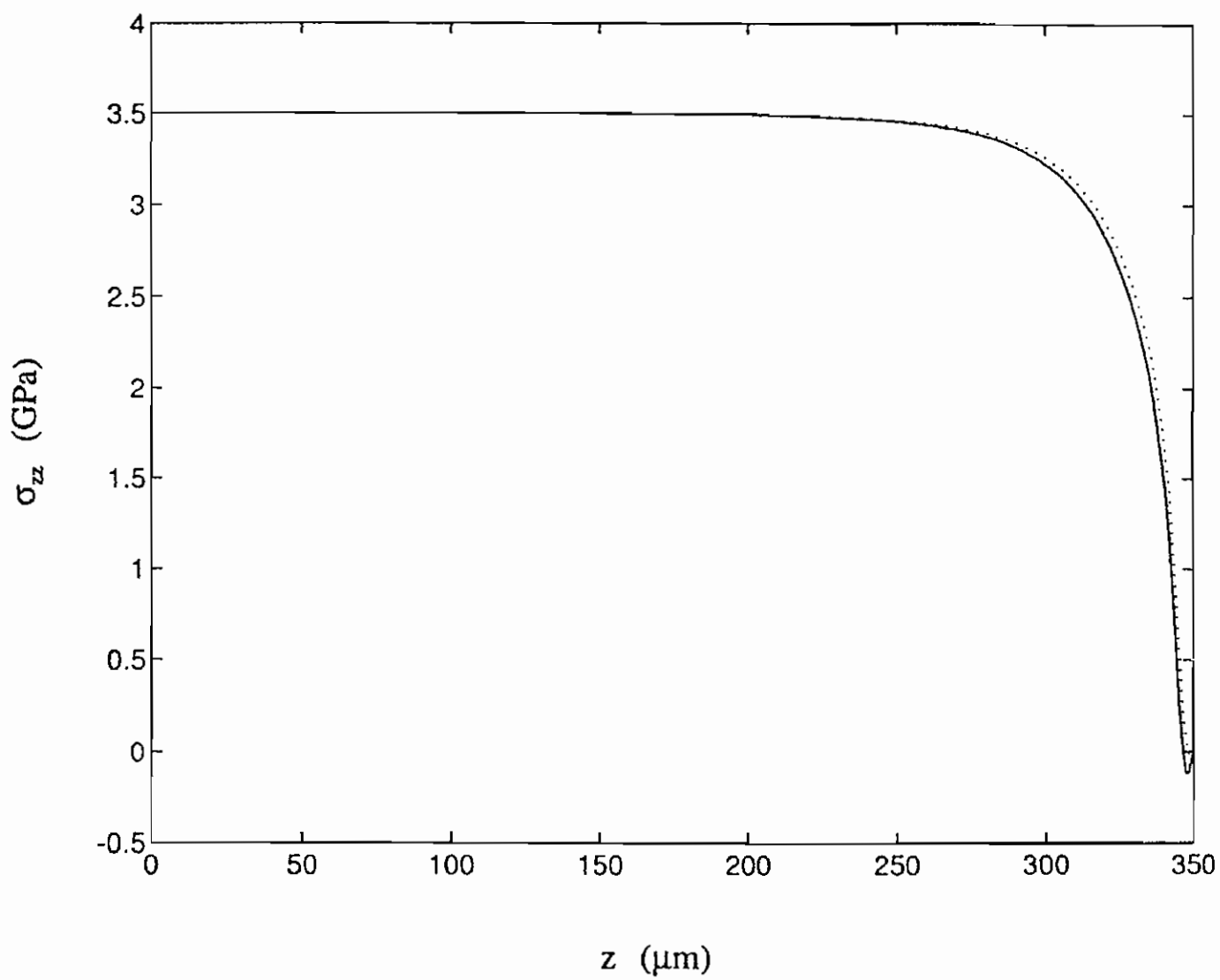


Fig.2a

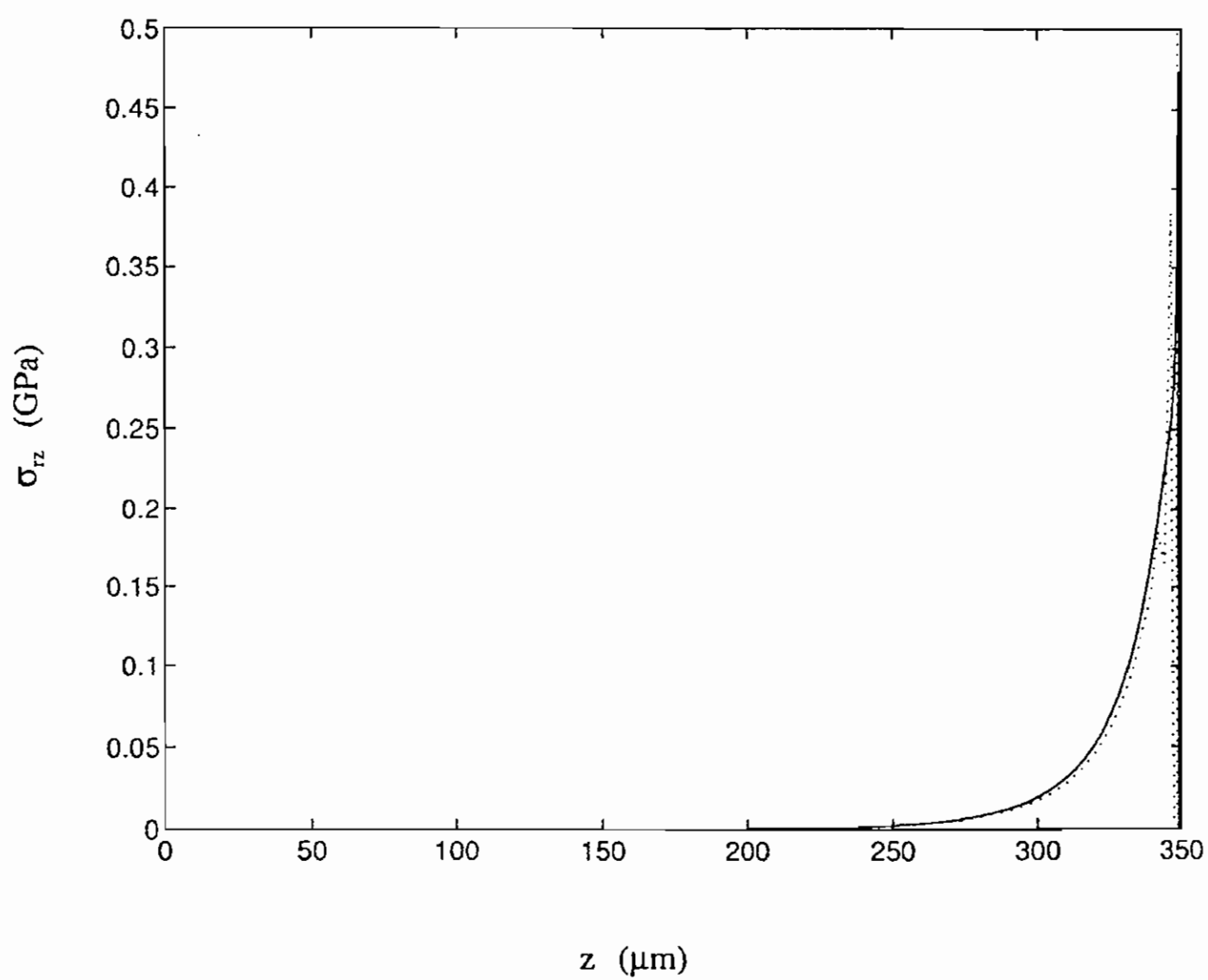


Fig.2b

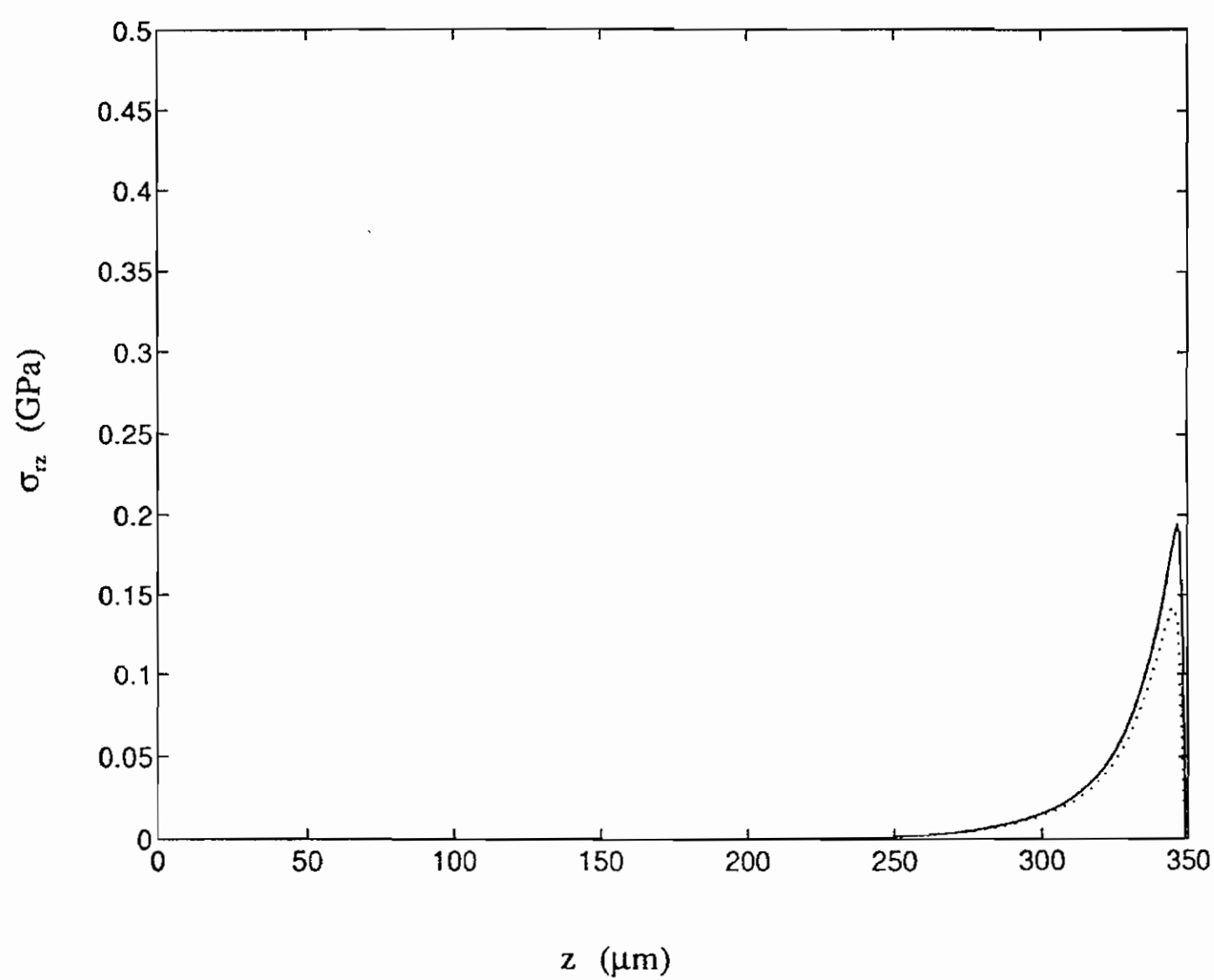


Fig.2c

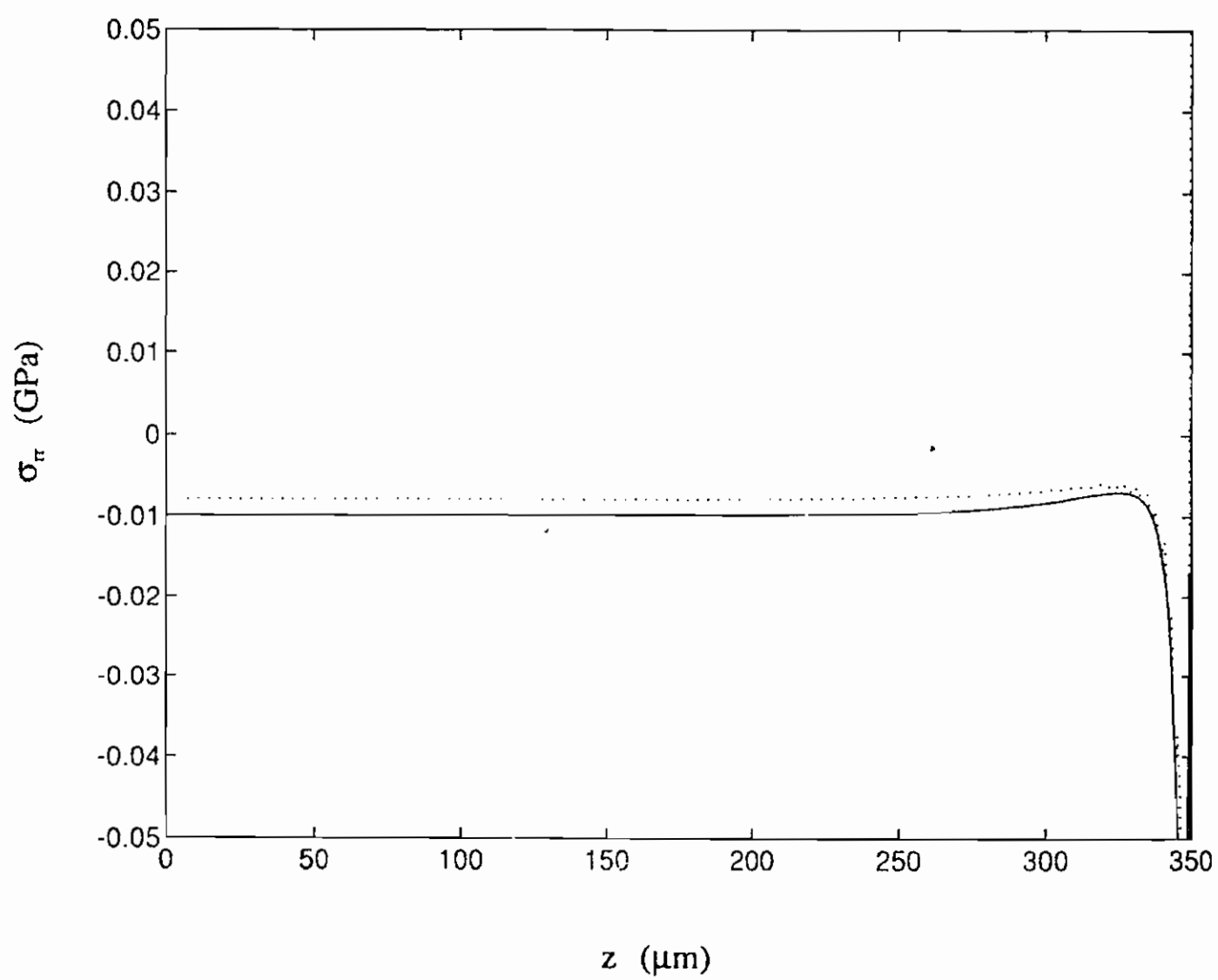


Fig.2d

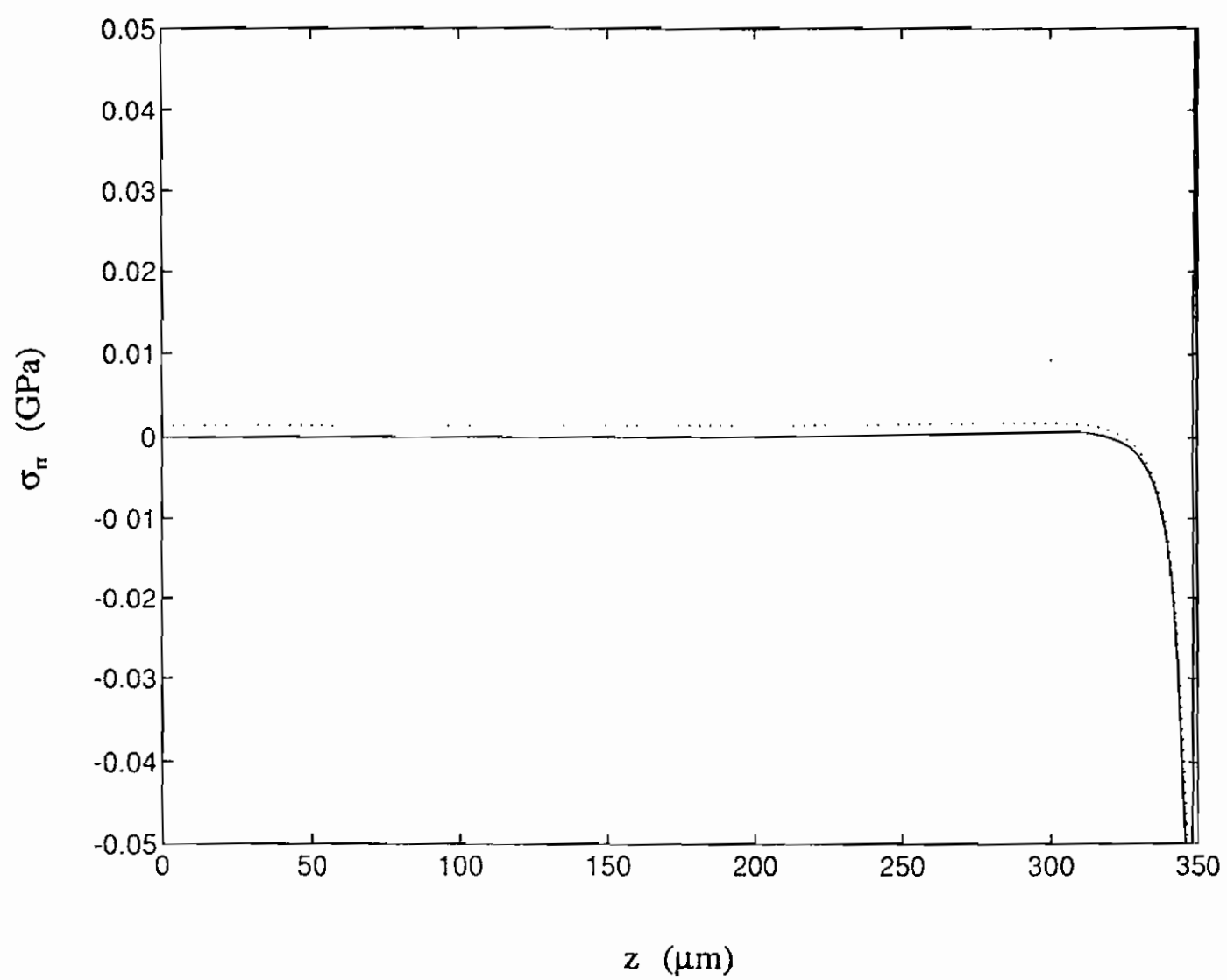


Fig.2e

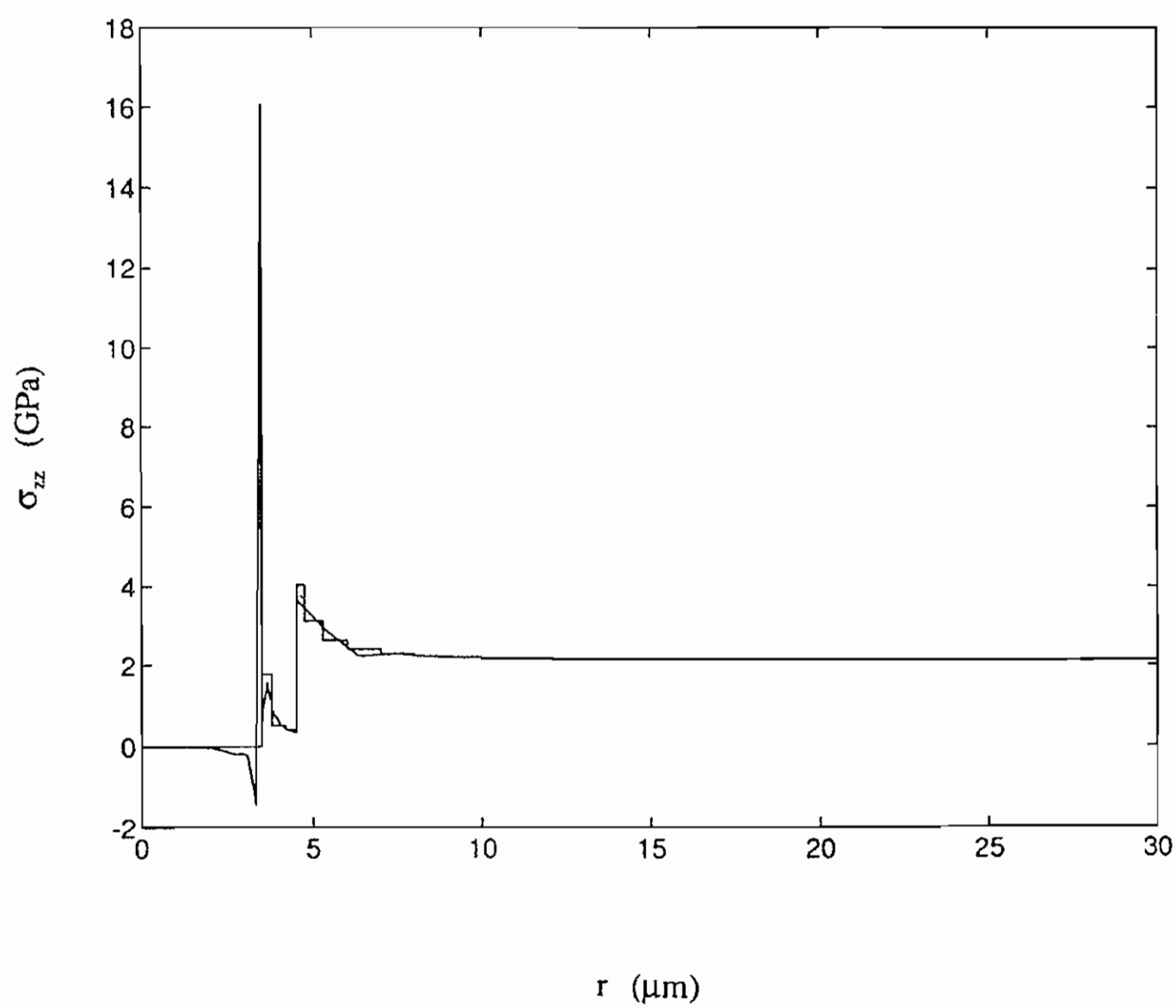


Fig.2f

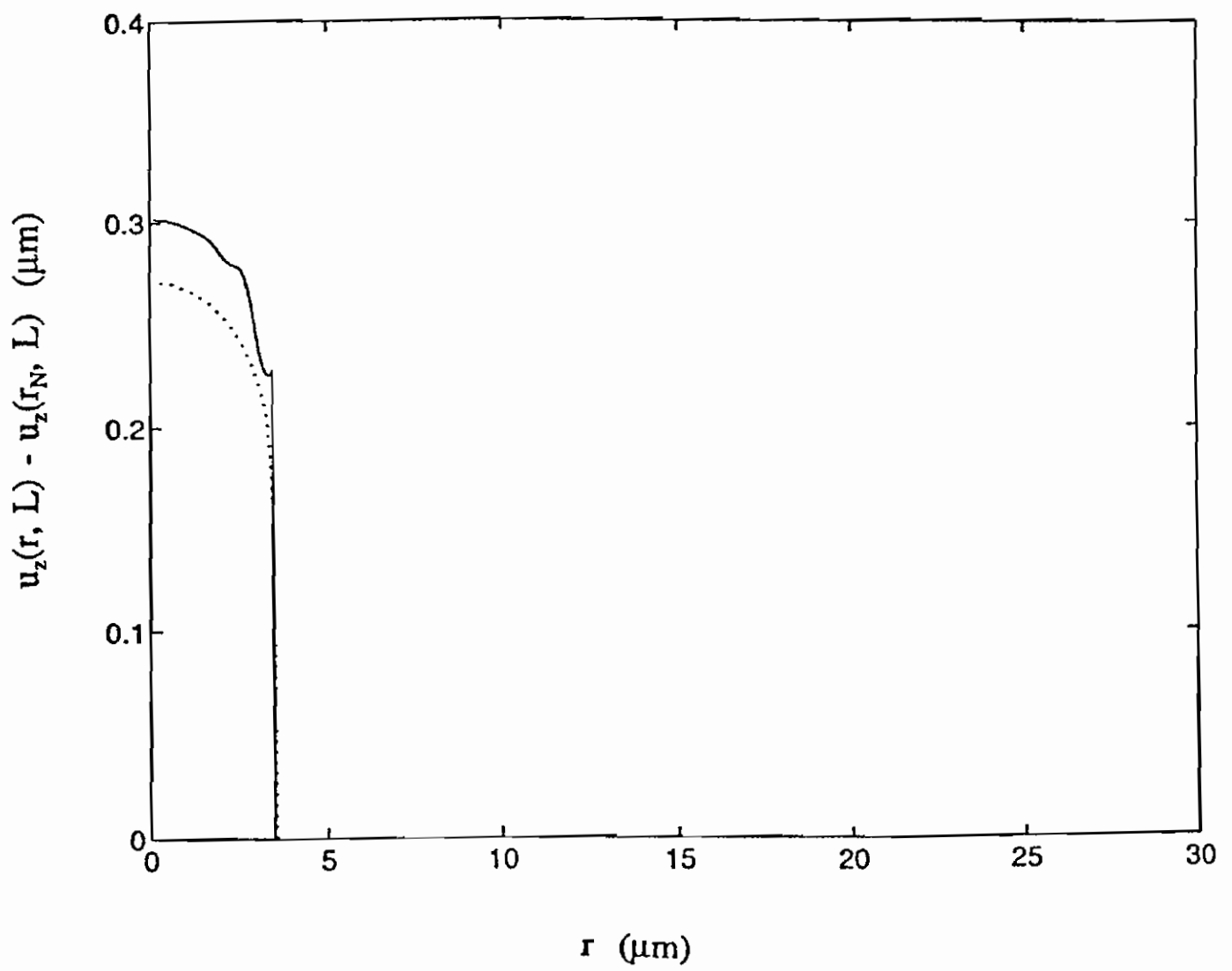


Fig.2g

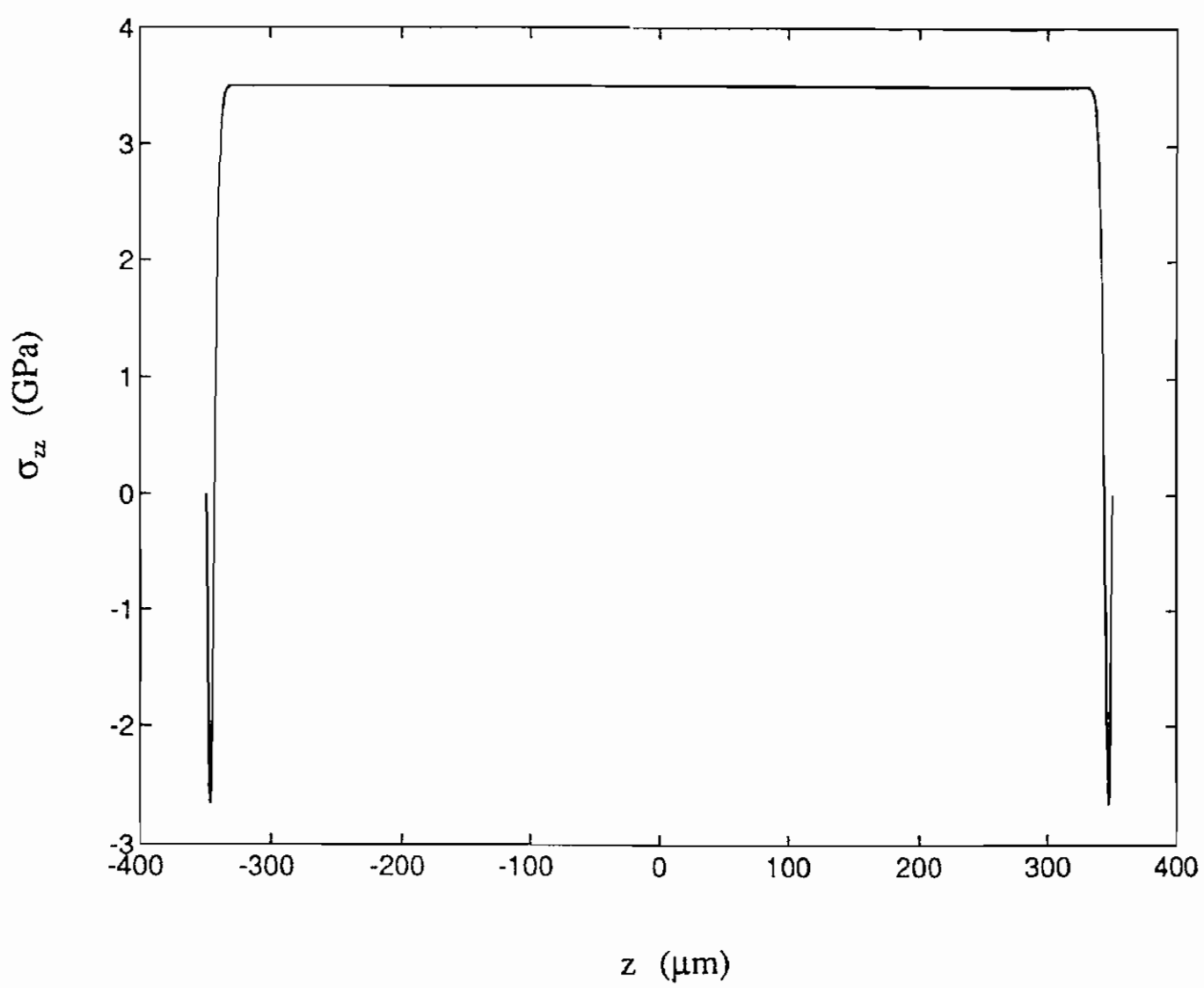


Fig.3

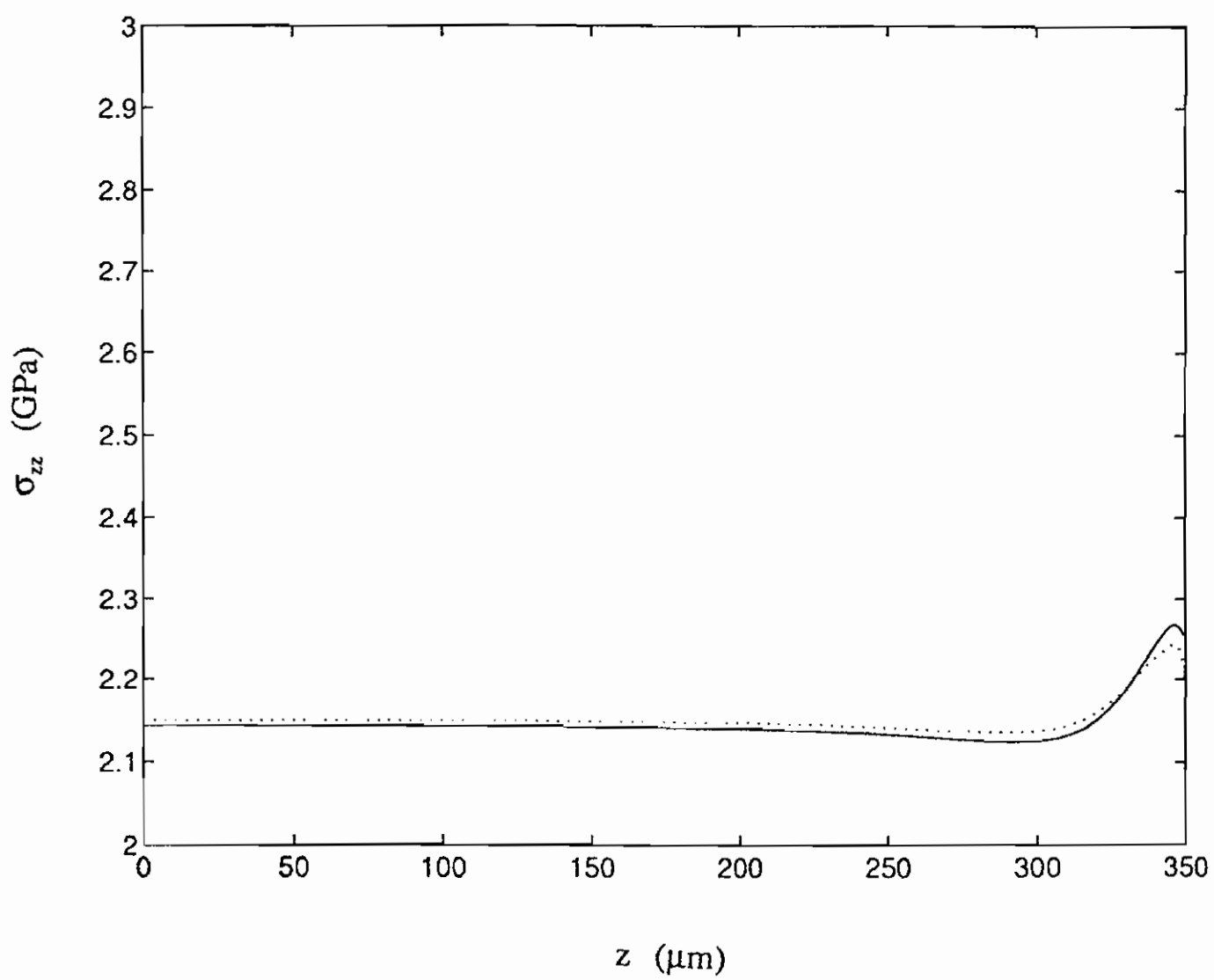


Fig.4

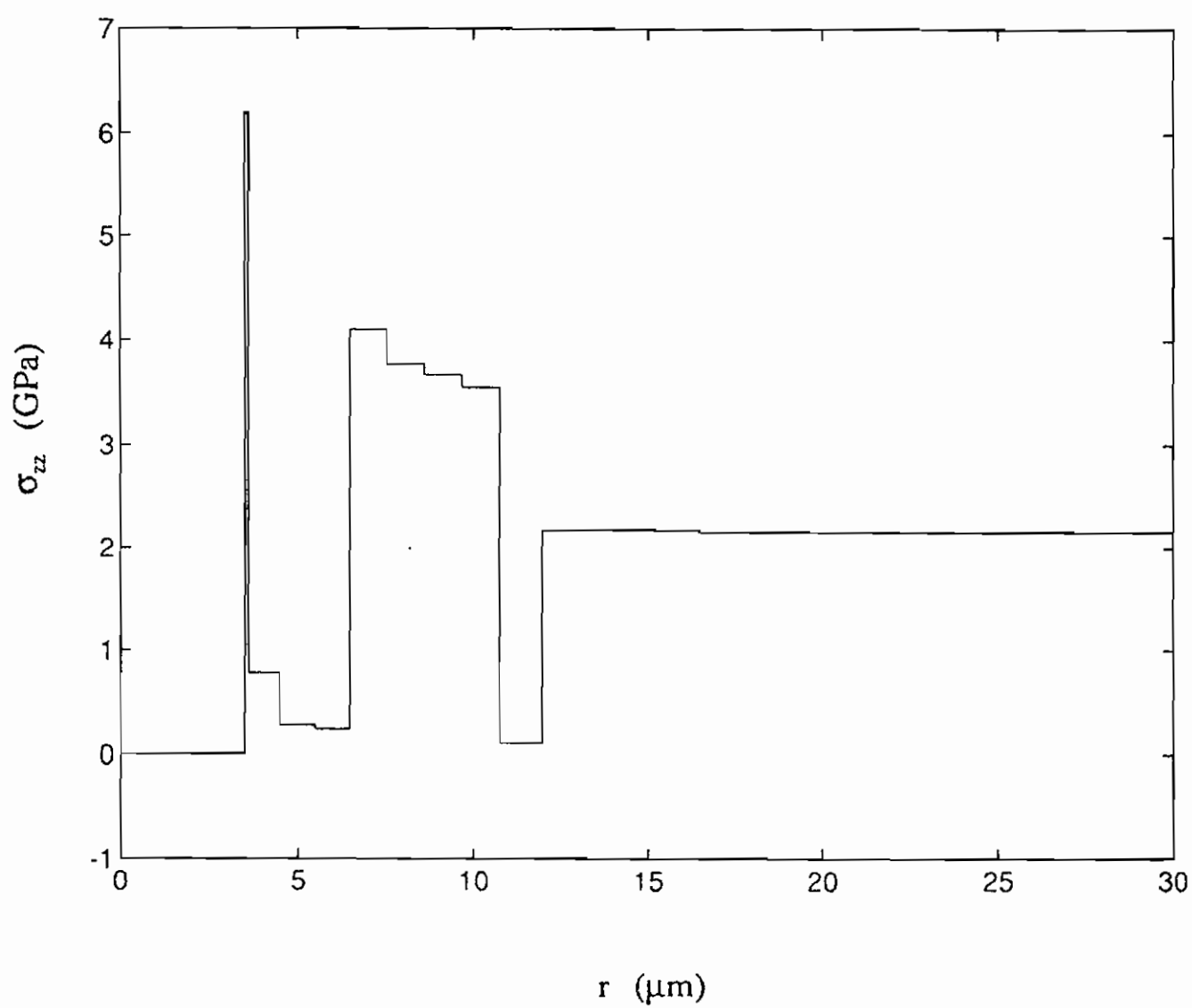


Fig.5