

REPORT

A Generalised Treatment of the Uncertainty in Calibration and Measurement of Vector- Indicating Microwave Reflectometers

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ABSTRACT

In this report we consider the uncertainty in calibration and measurement for vector-indicating microwave reflectometers.

The voltage reflection coefficient for a given measurement configuration is treated as a complex measurand. The uncertainty in measurement can be perceived as an uncertainty ellipse on each measured point in the complex plane with semi-axes and orientation depending on the errors in the real and imaginary parts of the reflection coefficient and their correlation.

In the first instance, we show how the system can be calibrated using the minimum number of standards (three) by solution of the resulting deterministic problem. We then show how the calibration uncertainty can be reduced by solving the over-determined system where more than three standards are considered. This solution involves the use of Generalised Distance Regression, as performed on complex values, to obtain the complex-valued calibration constants as used in the subsequent measurement process.

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1 Introduction

Many electrical measurements made at microwave frequencies are adversely affected by the fact that only a proportion of the incident (measuring) signal actually reaches the device under test (DUT). The reduction in signal is due to reflections from unwanted electrical discontinuities at the measurement reference plane, and elsewhere in the system. Such reflections induce both random and systematic errors into the measurements being performed and, in general, are apparent in all microwave measurements.

The problem is overcome by using a reflectometer to characterise the coefficient of reflection (defined as the ratio of reflected to incident voltage) for the given measurement configuration. This use enables a correction to be applied to allow for the amount of signal reflected from the DUT. Reflectometers therefore play a vital role in the majority of microwave measurement applications and thus warrant a detailed study of their performance.

This report presents such a study based on a generalised treatment of the uncertainty of calibration and measurement for a vector-indicating microwave reflectometer. Special consideration is given to the vector nature of all the quantities required to characterise the performance of the reflectometer for both calibration and measurement.

2 Reflectometer Theory

A reflectometer comprises a microwave signal source, detectors to sample the incident and reflected waveforms, and an assumed linear microwave network. It is often convenient to represent the reflectometer as a notionally perfect reflectometer connected to one side of a two-port network, the other side of which is connected to the DUT (Figure 1). The

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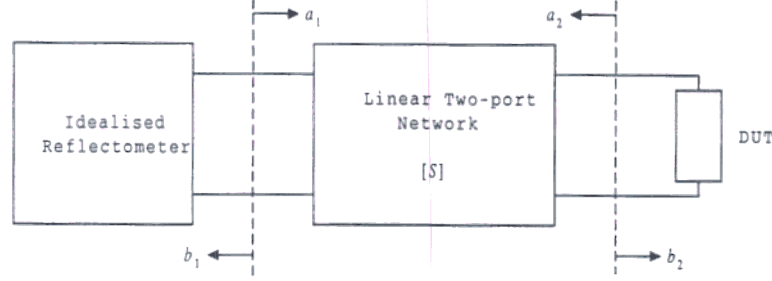


Figure 1: Schematic of Reflectometer.

two-port network then describes the characteristics of the non-ideal reflectometer. The idealised reflectometer indicates a vector, W , for the complex voltage reflection coefficient (VRC) Γ of the DUT.

Let

$$W = \frac{b_1}{a_1}, \quad (1)$$

where a_1 and b_1 are, respectively, the signals incident on, and reflected by, the microwave two-port network within the reflectometer.

Similarly, let

$$\Gamma = \frac{a_2}{b_2}, \quad (2)$$

where a_2 and b_2 are the signals reflected by, and incident on, the DUT.

The relation between the true value, Γ , of the reflection coefficient of the DUT and the value, W , indicated by the reflectometer is given by [1], and [2],

$$\Gamma = \frac{aW + b}{cW + 1}, \quad (3)$$

where a , b , and c are calibration constants that describe the internal network of the non-ideal reflectometer. The values of a , b , and c are determined by the calibration. The calibration of the reflectometer is achieved by connecting three or more calibration standards whose reflection coefficients Γ are assumed to be known. Thus, using equation (3),

$$\Gamma_i = \frac{aW_i + b}{cW_i + 1} \quad \text{for } i = 1, \dots, m, \quad (4)$$

where m is the number of standards, the Γ_i are the known reflection coefficients of the standards, and the W_i the corresponding reflection coefficients indicated by the reflectometer. By solving the system of equations (4) the calibration constants can be determined and any subsequent measurement can be corrected with the use of equation (3). Note that all variables are complex numbers, which we write in terms of their real and imaginary parts, $\Gamma = x + jy$, $W = u + jv$, $a = a_R + ja_I$, $b = b_R + jb_I$, and $c = c_R + jc_I$.

3 Calibration using three standards

This section considers the calibration of the reflectometer using three standards. This task is equivalent to setting up three equations of the form (4) for $i = 1, 2, 3$. This number of equations is the minimum sufficient to determine the three complex-valued calibration constants, a , b , and c .

Define the six-element vector $\gamma = (a_R, a_I, b_R, b_I, c_R, c_I)^T$, and the twelve-element vector, $s = (u_1, v_1, x_1, y_1, u_2, v_2, x_2, y_2, u_3, v_3, x_3, y_3)^T$. By substituting the real and imaginary parts of each term into equation (4), we obtain the three equations

$$\begin{aligned} (x_i + jy_i)(c_R + jc_I)(u_i + jv_i) + (x_i + jy_i) = \\ (a_R + ja_I)(u_i + jv_i) + (b_R + jb_I), \quad i = 1, 2, 3. \end{aligned} \quad (5)$$

Multiplying out the terms in equation (5) and then equating real and imaginary parts gives the linear matrix equation for the required coefficients,

$$\begin{pmatrix} u_1 & -v_1 & 1 & 0 & (y_1 v_1 - x_1 u_1) & (y_1 u_1 + x_1 v_1) \\ v_1 & u_1 & 0 & 1 & -(y_1 u_1 + x_1 v_1) & (y_1 v_1 - x_1 u_1) \\ u_2 & -v_2 & 1 & 0 & (y_2 v_2 - x_2 u_2) & (y_2 u_2 + x_2 v_2) \\ v_2 & u_2 & 0 & 1 & -(y_2 u_2 + x_2 v_2) & (y_2 v_2 - x_2 u_2) \\ u_3 & -v_3 & 1 & 0 & (y_3 v_3 - x_3 u_3) & (y_3 u_3 + x_3 v_3) \\ v_3 & u_3 & 0 & 1 & -(y_3 u_3 + x_3 v_3) & (y_3 v_3 - x_3 u_3) \end{pmatrix} \begin{pmatrix} a_R \\ a_I \\ b_R \\ b_I \\ c_R \\ c_I \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{pmatrix} \quad (6)$$

which may be denoted by

$$H\gamma = t, \quad (7)$$

or, in its implicit form,

$$g \quad H(s)\gamma - t(s) = 0. \quad (8)$$

It is now possible to determine the vector γ by solving the linear equations in (7). Due to the small nature of the problem, i.e. six-by-six, any stable method of solution, such as Gaussian elimination with partial pivoting or orthogonal factorisation, may be used.

In practice, both the observed values W_i and the assumed values given to the standards Γ_i have variances associated with their real and imaginary parts, together with a covariance between them. This information can be represented by a 2×2 covariance matrix in which the diagonal elements are the variances of the real and imaginary parts, and the off-diagonal contains the covariance. For practical purposes we will assume no correlation between Γ and W . However, the theory presented here is general enough to include this case in subsequent work.

We now analyse how these uncertainties affect the accuracy of the calibration.

To determine the covariance matrix we first differentiate equation (7) with respect to the components of s . For s_k we have

$$H \frac{\partial \gamma}{\partial s_k} + \frac{\partial H}{\partial s_k} \gamma = \frac{\partial t}{\partial s_k}. \quad (9)$$

Hence, letting

$$\frac{\partial \gamma}{\partial s} = \left[\frac{\partial \gamma}{\partial s_1}, \frac{\partial \gamma}{\partial s_{12}} \right], \quad \frac{\partial t}{\partial s} = \left[\frac{\partial t}{\partial s_1}, \frac{\partial t}{\partial s_{12}} \right]$$

$$H \frac{\partial \gamma}{\partial \mathbf{s}} = K,$$

where

$$K = \frac{\partial t}{\partial \mathbf{s}} - \left[\frac{\partial H}{\partial s_1} \gamma, \dots, \frac{\partial H}{\partial s_{12}} \gamma \right] \quad (11)$$

Now, in general, [3], if $\beta = \mathbf{f}(\alpha)$ and the covariance $V(\alpha)$ of α is given, the covariance of β is

$$V(\beta) = J_{\mathbf{f}} V(\alpha) J_{\mathbf{f}}^T, \quad (12)$$

where $J_{\mathbf{f}}$ is the Jacobian matrix of $\mathbf{f}$. It follows from equation (10) that the covariance of γ is

$$V(\gamma) = J V(\mathbf{s}) J^T, \quad (13)$$

where $V(\mathbf{s})$ is the covariance of $\mathbf{s}$ and J is the Jacobian matrix of γ with respect to $\mathbf{s}$ given by

$$HJ = K. \quad (14)$$

The quantities in the expression (11) for K are readily calculated. For instance, from equation (6),

$$\frac{\partial H}{\partial s_1} \equiv \frac{\partial H}{\partial u_1} = \begin{pmatrix} 1 & 0 & 0 & 0 & -x_1 & y_1 \\ 0 & 1 & 0 & 0 & -y_1 & -x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \frac{\partial t}{\partial \mathbf{s}_3} \equiv \frac{\partial t}{\partial \mathbf{x}_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (15)$$

Thus, having formed K , the solution of the linear equations (14) provides J . The required covariance matrix $V(\gamma)$ is then given by equation (13).

4 Calibration using more than three standards

In this section we show how to model the calibration of the reflectometer using more than the basic minimum number of three standards. Mathematically this more general calibration problem, which includes the use of three standards as a special case, can be stated as

$$\text{minimize } \mathbf{e}^T V^{-1} \mathbf{e} \quad (16)$$

with respect to $a_{\mathcal{R}}, a_{\mathcal{I}}, b_{\mathcal{R}}, b_{\mathcal{I}}, c_{\mathcal{R}}, c_{\mathcal{I}}$, and $\bar{u}_i, \bar{v}_i, \bar{x}_i, \bar{y}_i$, subject to the equality constraints,

$$\bar{\Gamma}_i = \frac{a \bar{W}_i + b}{c \bar{W}_i + 1} \quad i = 1, \dots, m, \quad (17)$$

where terms like $\bar{u}_i$ are the “true” values of u_i , and the vector $\mathbf{e}$ is the $4m$ -element vector comprising the error between measurement and true values for the inputs,

$$\mathbf{e} = (u_1 - \bar{u}_1, v_1 - \bar{v}_1, x_1 - \bar{x}_1, y_1 - \bar{y}_1, \dots, u_m - \bar{u}_m, v_m - \bar{v}_m, x_m - \bar{x}_m, y_m - \bar{y}_m)^T. \quad (18)$$

In (16), V denotes the covariance matrix $V(\mathbf{s})$, where now

$$\mathbf{s} = (u_1, v_1, x_1, y_1, \dots, u_m, v_m, x_m, y_m)^T$$

We may re-arrange the equality constraints in (17) as

$$\begin{aligned} \zeta(\gamma^*, u_i^*, v_i^*) &= x_i^*, \\ \eta(\gamma^*, u_i^*, v_i^*) &= y_i^*. \end{aligned} \quad (19)$$

These may be used to eliminate the terms in x_i^* and y_i^* so that the minimisation in equation (16), which is now unconstrained, is carried out with respect to $\mathbf{p}$, where $\mathbf{p} = (u_1^*, v_1^*, \dots, u_m^*, v_m^*, \gamma^T)$, with respect to $2m + 6$ variables.

This formulation is an example of a Generalised Distance Regression (GDR). Algorithms for solutions to problems of this kind may be found in [4] and [5]. A Gauss-Newton iteration technique can be used. It requires evaluation of the Jacobian matrix, J , for the system at each iteration [6]. The major computation in a stable implementation, [7], is the orthogonal factorisation $J = QR$ where Q is orthogonal and R is upper triangular.

We form the Cholesky decomposition of the covariance matrix $V = LL^T$. Then,

$$\mathbf{e}^T V^{-1} \mathbf{e} = (L^{-1} \mathbf{e})^T (L^{-1} \mathbf{e}) = \tilde{\mathbf{e}}^T \tilde{\mathbf{e}},$$

and thus the problem has been reduced to an unweighted least-squares problem in the elements of $\tilde{\mathbf{e}}$, where $\tilde{\mathbf{e}} = L^{-1} \mathbf{e}$. This is true in general, and only makes the assumption that V has a unique Cholesky factor, which is the case if no two elements of $\mathbf{s}$ are perfectly correlated.

To determine the uncertainties associated with the solution we proceed as follows. Let $\tilde{\mathbf{e}}$ denote the residuals $\tilde{\mathbf{e}}$ at the solution $\tilde{\mathbf{p}}$. Then [4],

$$J^T \tilde{\mathbf{e}} = \mathbf{0}, \quad (21)$$

where J is the Jacobian matrix at the solution for the transformed (unweighted) problem. But, linearised about $\tilde{\mathbf{p}}$,

$$\tilde{\mathbf{e}} = \tilde{\mathbf{e}} + J \delta \mathbf{p}$$

where $\delta \mathbf{p}$ is a perturbation to $\tilde{\mathbf{p}}$. Hence, from equations (21) and (22),

$$J^T \tilde{\mathbf{e}} = J^T J \delta \mathbf{p}, \quad (23)$$

and so,

$$\delta \mathbf{p} = (J^T J)^{-1} J^T \tilde{\mathbf{e}}. \quad (24)$$

Thus, again using the general result from equation (12),

$$V(\mathbf{p}) \equiv V(\delta \mathbf{p}) = (J^T J)^{-1} J^T V(\tilde{\mathbf{e}}) J (J^T J)^{-1} \quad (25)$$

But, since $V(\tilde{\mathbf{e}}) = I$, [8], it follows that

$$V(\mathbf{p}) = (J^T J)^{-1} \quad (26)$$

The determination of $V(\mathbf{p})$ can be obtained computationally from the upper-triangular matrix R corresponding to the Jacobian at the solution, using

$$V(\mathbf{p}) = (J^T J)^{-1} = ((QR)^T (QR))^{-1} = (R^T Q^T Q R)^{-1} = (R^T R)^{-1} = R^{-1} R^{-T}$$

Here, the fact that Q is orthogonal and hence $Q^T Q = I$ has been used.

The Jacobian J has structure (a zero-non zero pattern), but for the number m of standards likely to be used in practice there is no computational advantage in taking account of it. Should problems arise in which m is large, structure exploiting methods, [7], can be employed if necessary.

5 Obtaining measurement from observed values

We now have, in principle, all the information needed to obtain the reflection coefficient of an unknown DUT for both the determined and overdetermined case. We now consider its uncertainty. Assuming an observed value W_o , we use this, together with the determined calibration constants $\bar{a}$, $\bar{b}$, and $\bar{c}$ in the original calibration equation, (3),

$$\Gamma_o = \frac{\bar{a} W_o + \bar{b}}{\bar{c} W_o + 1} \quad (28)$$

The 2×2 covariance matrix for Γ_o can be calculated by writing equation (28) in the implicit form

$$\Gamma_o(\bar{c} W_o + 1) = \bar{a} W_o + \bar{b} \quad (29)$$

This can be split into real and imaginary parts

$$\begin{aligned} \Gamma_{\mathcal{R}}(c_{\mathcal{R}} W_{\mathcal{R}} - c_{\mathcal{I}} W_{\mathcal{I}} + 1) - \Gamma_{\mathcal{I}}(c_{\mathcal{I}} W_{\mathcal{R}} + c_{\mathcal{R}} W_{\mathcal{I}}) \\ (a_{\mathcal{R}} W_{\mathcal{R}} + a_{\mathcal{I}} W_{\mathcal{I}} + b_{\mathcal{R}}) &= 0, \\ \Gamma_{\mathcal{R}}(c_{\mathcal{I}} W_{\mathcal{R}} + c_{\mathcal{R}} W_{\mathcal{I}}) + \Gamma_{\mathcal{I}}(c_{\mathcal{R}} W_{\mathcal{R}} - c_{\mathcal{I}} W_{\mathcal{I}} + 1) \\ (a_{\mathcal{I}} W_{\mathcal{R}} + a_{\mathcal{R}} W_{\mathcal{I}} + b_{\mathcal{I}}) &= 0, \end{aligned}$$

where $\bar{a} = a_{\mathcal{R}} + j a_{\mathcal{I}}$, $\bar{b} = b_{\mathcal{R}} + j b_{\mathcal{I}}$, $\bar{c} = c_{\mathcal{R}} + j c_{\mathcal{I}}$, $W_o = W_{\mathcal{R}} + j W_{\mathcal{I}}$, and $\Gamma_o = \Gamma_{\mathcal{R}} + j \Gamma_{\mathcal{I}}$. The implicit version of this equation is

$$\mathbf{g}(\Gamma_o, \mathbf{q}) = \mathbf{0}, \quad (30)$$

where

$$\Gamma_o = \{\Gamma_{\mathcal{R}}, \Gamma_{\mathcal{I}}\}$$

and

$$\mathbf{q} = (\{W_{\mathcal{R}}, W_{\mathcal{I}}\}, \gamma^T)^T.$$

So the covariance matrix, $V(\Gamma_o)$, for the real and imaginary parts of the output can be found as follows.

The application of perturbation theory, with $\delta\Gamma_o$ and $\delta\mathbf{q}$, small perturbations in Γ_o and $\mathbf{q}$, respectively, gives

$$\frac{\partial \mathbf{g}}{\partial \Gamma_o} \delta\Gamma_o + \frac{\partial \mathbf{g}}{\partial \mathbf{q}} \delta\mathbf{q} = 0, \quad (31)$$

where $\frac{\partial \mathbf{g}}{\partial \Gamma_o}$ and $\frac{\partial \mathbf{g}}{\partial \mathbf{q}}$ are 2×2 partial-derivative matrices. Then,

$$\delta\Gamma_o = -C\delta\mathbf{q}, \quad (32)$$

where C is the 2×2 matrix that solves the linear equation

$$\frac{\partial \mathbf{g}}{\partial \Gamma_o} C = \frac{\partial \mathbf{g}}{\partial \mathbf{q}} \quad (33)$$

So, once again using the general result in equation (12),

$$V(\Gamma_o) = CV(\mathbf{q})C^T. \quad (34)$$

6 Results

To illustrate the performance of the presented technique, we give below some typical measurement results for the calibration and then measurement of a reflectometer system. An analysis of replicate measurements is used to obtain an estimate of W and its covariance matrix for the calibration standards in the case where $m = 3$. In the cases where four and five calibration standards are used, calibration information has been provided in the form of simulated data. The uncertainties in the simulated data have been set to the upper bound of the values of those for the measured data. It is intended to include measured data in the extra calibration points in the near future when data of this kind becomes available.

For simplicity, the uncertainty in the true values of the standards is not considered here. This has been achieved in practice by setting the uncertainty in each of the “true” values for the calibration standards, Γ_i to a negligibly small value. These values are used to obtain the calibration coefficients for the system, together with their covariance matrix. The calibration data and its uncertainty, together with assumed uncertainty on measured values, is then used in the measurement process to obtain values for the true reflection coefficients of the DUTs and their uncertainties.

In Figure 2.1 the measured values together with typical uncertainty ellipses for four points in the complex plane are shown. These correspond to four measured values of W together with uncertainties in the real and imaginary parts. The ellipses, assuming a Gaussian distribution for the error, represent 95% confidence contours, scaled for presentational purposes by an amplification factor of 10^4 .

In Figure 2.2 we show the corresponding “true” values and their uncertainty of the four reflection coefficients measured in Figure 2.1. In this case, the system is calibrated using three realisable calibration items with values $\Gamma = (-1, 0, 1)$.

In Figure 2.3 the same values are plotted with uncertainty when the system has been calibrated using four calibration items, with values $\Gamma = (-1, 0, 1, +j)$.

Finally in Figure 2.4 we show the values and their uncertainty calculated using five calibration items with values $\Gamma = (-1, 0, 1, +j, -j)$.

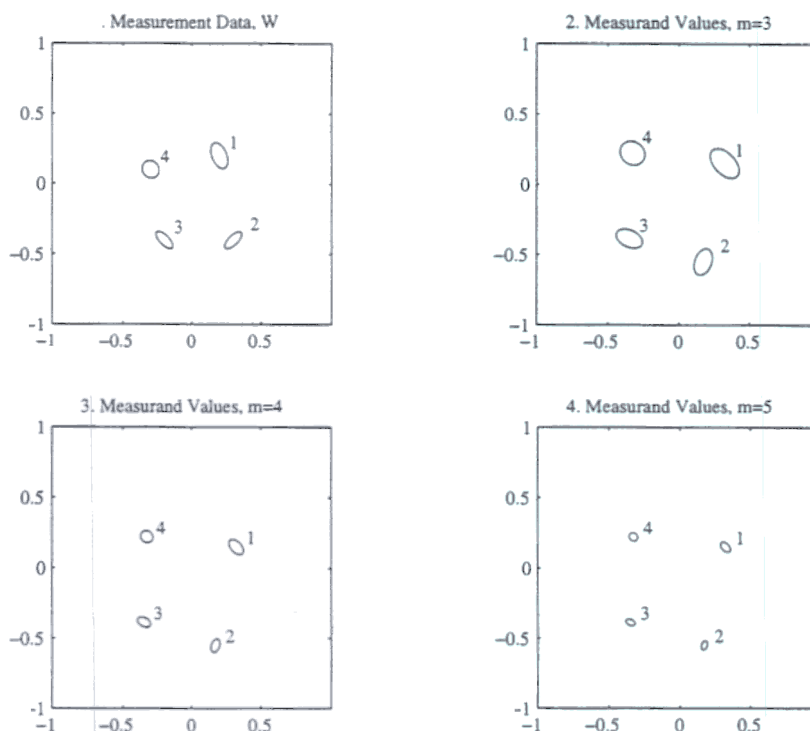


Figure 2: Measurement results showing the effect, for four measured values of W , of calibrating a reflectometer using $m = 3, 4$, and 5 standards. The ellipses represent 95% confidence contours, assuming Gaussian errors. They are scaled for presentational purposes by an amplification factor of 10^4 .

7 Conclusions

A method has been presented which treats the errors associated with vector-indicating microwave reflectometers in terms of the covariance matrix of the real and imaginary parts of the complex quantities involved. The classical three-standard calibration method has been extended to allow calibration using more than three standards. The improvement in performance of the reflectometer has been demonstrated by presenting typical measured results achievable for the system. It has been shown that for five calibration standards, where $\Gamma = (-1, 0, 1, -j, j)$ the uncertainty in the measurement has decreased by a significant amount from that of the conventional three-standard technique. It is expected that additional standards would further improve the measurement accuracy.

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