

A NEW MODEL OF FATIGUE-CREEP DEFORMATION

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ABSTRACT

This report describes an approach to the prediction of the combined fatigue-creep behaviour of materials without damage that is based on the concept of internal state variables. The approach leads to the decomposition of the strain field into elastic, plastic, reversible anelastic and irreversible creep strain components which are defined for multi-axial states of stress.

The model has been implemented as a user-friendly software system known as LOOP+ which enables users, on defining the parameters that occur in the model, to investigate the effects of their selection on multi-axial materials behaviour. Examples are given of the various types of materials behaviour that can be predicted by the model

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1. Introduction

A feature of engineering components, that separates their behaviour from that of coupons tested in a laboratory, is the non-uniform stress distributions that arise from geometrical features, e.g. a hole in a plate is perhaps the simplest example. Fatigue and creep deformation mechanisms lead to changes in the material state which vary from point to point in the component. The deformation arising will lead to load transfer in engineering components where regions of larger stress deform more thus lowering the local stress. The load shed locally is transferred to other parts of the component which are then subject to increased deformation. Understanding the deformation behaviour of materials subject to fatigue and creep in multi-axial loading conditions is a very critical issue for the development of reliable design methodology. Load transfer mechanisms arising in non-uniform stress fields also have an important affect on localised damage development that eventually leads to catastrophic failure. This report is concerned only with the development of models of deformation arising from the operation of metals subject to fatigue/creep loading which is assumed to be multi-axial in nature.

A major requirement, concerning the use of metallic components in an engineering environment, is the development of an understanding of the likely failure mechanisms that could be encountered in service conditions. While strength is a performance measure of prime importance to engineers, an almost equally important performance measure is the expected lifetime of an engineering component. Strength testing assesses the resistance of a component to the effects of peak applied loads that could arise when the component is operated in an overload situation. Strength is usually determined by the maximum size of defect (e.g. micro-crack or pore) that becomes critical in a component subject to progressively increasing loading. Strength is a parameter that is well known to be statistically distributed. Lifetime is often governed by sub-critical defect growth arising from fatigue loading (i.e. cyclic loading at levels that do not initially cause catastrophic failure but which cause the growth of sub-critical defects to the point of criticality because of fatigue mechanisms) or creep deformation (i.e. loading at elevated temperatures and fixed loads where sub-critical defects progressively grow to the point of criticality i.e. catastrophic failure). For components operating under service loads at elevated temperatures both fatigue and creep operate together and interact

The defects in the component will grow during service at rates determined by the local stress field which is usually triaxial in nature, contrasting sharply with the uniaxial (or sometimes biaxial) testing carried out in the laboratory. Defects will grow faster in those regions of higher stress leading to localised damage states that are not uniform. While this is an important phenomenon affecting the lifetime of engineering components, it is beyond the scope of the current investigation. However, an approach that will deal with damage phenomena is the subject known as continuum damage

mechanics (see for example Chaboche [1,2]). The approach is based on the application of the principles of continuum thermodynamics for an equation of state for the material that involves a scalar damage parameter which is an attempt to account for the effects of material damage on material properties, and to enable the prediction of material failure. The damage parameter is treated as though it were an internal state variable. McCartney [3] has developed an approach to internal state variables and their effect on material behaviour in the absence of damage. Their role was shown to be associated with anelastic deformation that will be considered in detail in this report. It is emphasised that the internal state variable and continuum damage mechanics approaches are phenomenological in nature as they do not attempt to predict the microstructure and associated local stress and displacement fields in a metal alloy, but they are concerned only with the macroscopic stress and displacement fields that are governed by constitutive laws which are derived from the application of the principles of continuum thermodynamics. A consequence of this approach is that the constitutive laws proposed, when expressed in their most general form, involve materials parameters that have to be either measured experimentally or inferred from numerical predictions using the models.

This report will describe the theoretical basis of the internal state variable approach, and show how it can be used to define a material model that exhibits combined elastic-plastic and creep behaviour for multi-axial time dependent stress states. The report will also describe briefly the implementation of the model as a user-friendly software system known as LOOP+ . Example predictions are given that illustrate the types of material behaviour that can be modelled. Such examples also enable users of LOOP+ to check that their software is working correctly.

2. General theory for continuum thermodynamics

In terms of rectangular Cartesian coordinates x_j , $j = 1, 2, 3$ and the time t , the local form for the energy balance equation for infinitesimal deformations has the well known form see for example [3]

$$\frac{\partial(\rho_0 \epsilon)}{\partial t} = - \frac{\partial h_i}{\partial x_i} + \sigma_{kl} \frac{\partial e_{kl}}{\partial t} + \rho_0 r, \quad (1)$$

where ρ_0 is the uniform initial density of the material, ϵ is the specific internal energy, h_i is the i^{th} component of the heat flux vector, σ_{kl} are the components of the stress tensor, e_{kl} are the corresponding components of the strain tensor, and where r is the rate of energy supply per unit mass (e.g. from electrical heating if present). This term will be used to ensure that precise isothermal conditions are achieved by an imagined local addition or extraction of heat. Such a situation cannot be realised in practice as heat is gained or lost through the external surface by means of the heat flux

vector. It follows therefore that precise isothermal conditions cannot occur during the deformation of real materials. However, if the deformation is slow enough the temperature will be almost uniform everywhere, and the system will behave as though heat were being extracted locally through the imagined heat supply term r . For infinitesimal deformations the strain tensor is defined in terms of the components u_k as follows

$$e_{kl} = \frac{1}{2} \left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right) = e_{lk} . \quad (2)$$

In order to develop a theory of fatigue-creep interactions, an equation of state must be specified. Before specifying the general form of the equation of state on which subsequent modelling will be based, it is useful to discuss in some detail the various contributions to deformation into which the strain tensor may be decomposed. In order to explain the strain concepts clearly, it is useful to define the nature of the various strain components with reference to a homogeneous sample of a perfect cubic crystal. At a particular reference temperature the displacement function for the lattice points is zero everywhere in the crystal. If the crystal is homogeneously instantaneously loaded and/or the temperature changed, assuming that inertia effects may be neglected, the strain response may be expressed as a linear function of the stress field and the temperature change. The constants appearing in the linear relationship are identified with the elastic and thermal expansion coefficients of the crystal. Deformations arising in such a way are considered to define the elastic component of the strain tensor e_{kl}^E .

If the load that has been applied is now held fixed, and if the temperature change is large enough, vacancy diffusion within the crystal is then possible. The migration of vacancies is balanced by a net material flow in the opposite direction and this is well known to lead to shape changes of the crystal where the crystal extends on the loaded surface and recedes on the unloaded surfaces leading to a time dependent creep strain in the crystal. Such a mechanism leads to permanent crystal strains which are not recoverable when the stress is removed and the temperature is returned to the reference value. Such permanent deformations define the plastic component of the strain tensor e_{kl}^P . Plastic strains can also arise from instantaneous slip processes within a crystal associated with dislocation generation and motion

Because of interaction effects between both vacancy diffusion and dislocation motion, a third type of deformation occurs which is time dependent defining an anelastic component of the strain tensor e_{kl}^A . This strain has the property that when a crystal has been loaded for a sufficient time a component of strain progressively develops, and would gradually diminish to zero if the load was removed and the temperature held fixed.

The total strain tensor e_{kl} may be written

$$e_{kl} = e_{kl}^E + e_{kl}^A + e_{kl}^P \quad (3)$$

It will be useful for subsequent developments to define a reversible strain tensor as follows

$$e_{kl}^R = e_{kl}^E + e_{kl}^A, \quad (4)$$

so that the total strain may be written

$$e_{kl} = e_{kl}^R + e_{kl}^P \quad (5)$$

The reversible strain tensor is fully reversible following the removal of the applied load, although it may take some time for the material to fully recover. The strain that is instantaneously recovered is the elastic strain, while the strain whose recovery is time dependent is known as the anelastic strain.

A useful approach is to assume that the specific internal energy has the form

$$\epsilon = \epsilon(\eta, e_{kl}^R, s_{kl}^1, s_{kl}^2, \dots, s_{kl}^N), \quad (6)$$

where η is the specific entropy and where s_{kl}^I , $I = 1 \dots N$, denote a set of symmetric tensorial internal state variables. The reversible strain tensor has been used as a state variable in preference to the total strain. The example of the creep deformation of a crystal structure by vacancy diffusion indicates why this is the correct choice. The energy stored in the crystal is dependent upon the lattice deformations, and not upon the permanent shape changes that contribute to the total strain when the crystal is modelled as a continuum. The internal state variables appearing in (6) cannot be defined in physical terms at this point of the development of the model.

The incremental form of the equation of state is written

$$d(\rho_0 \epsilon) = \theta d(\rho_0 \eta) + \sigma_{kl} de_{kl}^R + \sum_{I=1}^N \mu_{kl}^I ds_{kl}^I, \quad (7)$$

where θ denotes the thermodynamic temperature (i.e. the absolute temperature which is always positive) defined by

$$\theta = \frac{\partial \epsilon}{\partial \eta}, \quad (8)$$

where the stress tensor is given by

$$\sigma_{kl} = \frac{\partial(\rho_0 \epsilon)}{\partial e_{kl}^R} = \sigma_{lk}, \quad (9)$$

and where $\mu_{kl}^I = \dots$, denoting thermodynamic potentials associated with the internal state variables, are defined by

$$\mu_{kl}^I = \frac{\partial(\rho_0 \epsilon)}{\partial s_{kl}^I} = \mu_{lk}^I, \quad I = 1 \dots N$$

The relations (1), (5) and (7) imply that

$$\theta \frac{\partial(\rho_0 \eta)}{\partial t} = - \frac{\partial h_i}{\partial x_i} + \sigma_{kl} \frac{\partial e_{kl}^P}{\partial t} - \sum_{I=1}^N \mu_{kl}^I \frac{\partial s_{kl}^I}{\partial t} + \rho_0 r,$$

which is now rearranged into the form of an entropy balance equation as follows

$$\frac{\partial(\rho_0 \eta)}{\partial t} = - \frac{\partial}{\partial x_i} \left(\frac{h_i}{\theta} \right) + \Delta + \frac{\rho_0 r}{\theta},$$

where Δ is the local rate of entropy production per unit volume resulting from irreversible thermodynamic processes defined by

$$\Delta = h_i \frac{\partial}{\partial x_i} \left(\frac{1}{\theta} \right) + \frac{1}{\theta} \left[\sigma_{kl} \frac{\partial e_{kl}^P}{\partial t} - \sum_{I=1}^N \mu_{kl}^I \frac{\partial s_{kl}^I}{\partial t} \right]$$

The second law of thermodynamics is satisfied if $\Delta \geq 0$ (Clausius-Duhem inequality) for all possible local states of the system.

In subsequent developments it is useful to impose at any time t , precise isothermal conditions by regulating the rate of heat supply r so that the temperature θ is everywhere uniform but which may not be constant in time. Under such conditions the relation (13) reduces to

$$\Delta = \frac{1}{\theta} \left[\sigma_{kl} \frac{\partial e_{kl}^P}{\partial t} - \sum_{I=1}^N \mu_{kl}^I \frac{\partial s_{kl}^I}{\partial t} \right], \quad (14)$$

and () leads to the following expression for r

$$\rho_0 r = \theta \frac{\partial(\rho_0 \eta)}{\partial t} - \sigma_{kl} \frac{\partial e_{kl}^P}{\partial t} + \sum_{I=1}^N \mu_{kl}^I \frac{\partial s_{kl}^I}{\partial t} \quad (15)$$

The relation (15) determines the local value needed for the rate of heat supply r in order that the temperature θ is everywhere uniform at any time. When the temperature θ is held fixed the R.H.S. of (15) corresponds to the negative of the local rate of energy dissipation that is required in order to maintain isothermal conditions.

On substituting (15) in (1) using (5) in order to eliminate r , it then follows that the local form for the isothermal energy balance equation has the form

$$\frac{\partial(\rho_0 \psi)}{\partial t} = \sigma_{kl} \frac{\partial e_{kl}^R}{\partial t} + \sum_{I=1}^N \mu_{kl}^I \frac{\partial s_{kl}^I}{\partial t} ,$$

where ψ is the specific Helmholtz free energy defined by

$$\psi = \varepsilon - \theta \eta \quad (17)$$

It follows from (7) and (17) that

$$d(\rho_0 \psi) = -\rho_0 \eta d\theta + \sigma_{kl} de_{kl}^R + \sum_{i=1}^N \mu_{kl}^I ds_{kl}^I ,$$

implying that an equation of state could also be specified of the form

$$\psi = \psi(\theta, e_{kl}^R, s_{kl}^1, s_{kl}^2, \dots, s_{kl}^N) , \quad (19)$$

such that

$$\eta = -\frac{\partial \psi}{\partial \theta} , \quad (20)$$

$$\sigma_{kl} = \frac{\partial(\rho_0 \psi)}{\partial e_{kl}^R} = \sigma_{lk} , \quad (21)$$

$$\mu_{kl}^I = \frac{\partial(\rho_0 \psi)}{\partial s_{kl}^I} = \mu_{lk}^I , \quad I=1 \dots N . \quad (22)$$

The specific Gibbs free energy ϕ is now introduced defined by

$$\phi = \psi - \frac{1}{\rho} \sigma_{kl} e_{kl}^R , \quad (23)$$

so that from (18)

$$d(\rho_0 \phi) = -\rho_0 \eta d\theta - e_{kl}^R d\sigma_{kl} + \sum_{i=1}^N \mu_{kl}^I ds_{kl}^I , \quad (24)$$

implying that an equation of state could be specified of the form

$$\phi = \phi(\theta, \sigma_{kl}, s_{kl}^1, s_{kl}^2, \dots, s_{kl}^N) , \quad (25)$$

such that

$$\eta = -\frac{\partial \phi}{\partial \theta} , \quad (26)$$

$$e_{kl}^R = -\frac{\partial(\rho_0 \phi)}{\partial \sigma_{kl}} = e_{lk}^R , \quad (27)$$

$$\mu_{kl}^I = \frac{\partial(\rho_0 \phi)}{\partial s_{kl}^I} = \mu_{lk}^I , \quad I=1 \dots N \quad (28)$$

The relation (16), valid for isothermal conditions, may now be written

$$\frac{\partial(\rho_0\phi)}{\partial t} = -e_{kl}^R \frac{\partial\sigma_{kl}}{\partial t} + \sum_{I=1}^N \mu_{kl}^I \frac{\partial s_{kl}^I}{\partial t}$$

3. Equation of state

Before further progress can be made it is necessary to impose the form of the equation of state. First of all, it is necessary to define the reference state of the system which is characterised by zero values of the stress tensor, all strain tensors, and all internal state variables. The reference temperature is assumed to have the value q_0 . As stress and temperature are convenient state variables to use in practice, an equation for the Gibbs free energy is assumed of the quadratic form

$$\begin{aligned} \rho_0\phi = & \rho_0\phi(\theta) - \alpha(\theta - \theta_0)\tilde{\sigma} - \frac{1-2\nu}{6E}\tilde{\sigma}^2 - \frac{1+\nu}{2E}\hat{\sigma}_{ij}\hat{\sigma}_{ij} \\ & + \frac{1}{3}\sum_{I=1}^N \left[A_I \tilde{s}^I + \frac{1}{2}B_I(\tilde{s}^I)^2 \right] + \sum_{I=1}^N \left[C_I \hat{\sigma}_{ij}\hat{s}_{ij}^I + \frac{1}{2}D_I \hat{s}_{ij}^I \hat{s}_{ij}^I \right] \end{aligned}$$

where E is Young's modulus, ν is Poisson's ratio, α is the thermal expansion coefficient, and where A_I , B_I , C_I and D_I , $I = 1 \dots N$ are material constants that may be temperature dependent but this will not be indicated explicitly. In (30) the tensors have been split into their hydrostatic and deviatoric components such that for any tensor a_{ij} the hydrostatic component is $a_{kk} = \tilde{a}$ and the deviatoric component is $\hat{a}_{ij}$ such that

$$a_{ij} = \hat{a}_{ij} + \frac{1}{3}\tilde{a}\delta_{ij} \quad (31)$$

where δ_{ij} is the Kronecker delta symbol having the value 1 whenever $i = j$, and the value zero otherwise. The form of (30) selected is the most general quadratic function of the stress and internal state variables that is possible for an isotropic material for the special case where there is no interaction between the various internal state variables whose reference value is zero (leading to the absence in (30) of linear terms involving only the internal state variables). Interaction effects could be included by introducing cross products involving the various types of internal state. It is always expected that (30) will be valid for small departures from the reference state. On using (26-28) it follows that

$$\rho_0 \eta = -\rho_0 \phi_0(\theta) + \alpha \tilde{\sigma} + \frac{\partial}{\partial \theta} \left(\frac{1-2\nu}{6E} \right) \tilde{\sigma}^2 + \frac{\partial}{\partial \theta} \left(\frac{1+\nu}{2E} \right) \hat{\sigma}_{ij} \hat{\sigma}_{ij} \quad (32)$$

$$\frac{1}{3} \sum_{I=1}^N \left[\frac{\partial A_I}{\partial \theta} \tilde{\sigma} \tilde{s}^I + \frac{1}{2} \frac{\partial B_I}{\partial \theta} (\tilde{s}^I)^2 \right] - \sum_{I=1}^N \left[\frac{\partial C_I}{\partial \theta} \hat{\sigma}_{ij} \hat{s}_{ij}^I + \frac{1}{2} \frac{\partial D_I}{\partial \theta} \hat{s}_{ij}^I \hat{s}_{ij}^I \right]$$

$$\begin{aligned} \tilde{e}^R &= 3\alpha(\theta - \theta_0) + \frac{1-2\nu}{E} \tilde{\sigma} - \sum_{I=1}^N A_I \tilde{s}^I \\ \hat{e}_{kl}^R &= \frac{1+\nu}{E} \hat{\sigma}_{kl} - \sum_{I=1}^N C_I \hat{s}_{kl}^I \end{aligned} \quad (33)$$

$$\tilde{\mu}^I = A_I \tilde{\sigma} + B_I \tilde{s}^I \quad = 1 \dots N ,$$

$$\hat{\mu}_{kl}^I = C_I \hat{\sigma}_{kl} + D_I \hat{s}_{kl}^I , \quad = 1 \dots N$$

4. Modelling creep deformation

As already mentioned above the second law of thermodynamics is satisfied whenever the Clausius-Duhem inequality $\Delta \geq 0$ is obeyed where the rate of entropy production is given by (13) for non-isothermal conditions, and by (14) for isothermal conditions. All constitutive laws of material behaviour must be consistent with this inequality. Since the thermodynamic temperature θ is never negative the second law is obeyed whenever

$$\sigma_{kl} \frac{\partial e_{kl}^P}{\partial t} - \sum_{I=1}^N \mu_{kl}^I \frac{\partial s_{kl}^I}{\partial t} \geq 0 \quad (35)$$

for all possible states of the system.

It is now assumed, in keeping with the literature, that plastic strain does not lead to a change of volume so that for infinitesimal deformations $e_{kk}^P = 0$. The inequality (35) may then be decomposed as follows

$$\sum_{I=1}^N \tilde{\mu}^I \frac{\partial \tilde{s}^I}{\partial t} \leq 0 , \quad \hat{\sigma}_{kl} \frac{\partial \hat{e}_{kl}^P}{\partial t} - \sum_{I=1}^N \hat{\mu}_{kl}^I \frac{\partial \hat{s}_{kl}^I}{\partial t} \geq 0 . \quad (36)$$

It is now postulated that

$$\frac{\partial \hat{e}_{kl}^P}{\partial t} = \Lambda \omega \left(\frac{\bar{\sigma}}{E} \right)^{2m} \hat{\sigma}_{kl} ,$$

where the parameter $\bar{\sigma}$ is one invariant of the deviatoric stress known as the effective stress that is defined by

$$\bar{\sigma} = \sqrt{\frac{3}{2} \hat{\sigma}_{ij} \hat{\sigma}_{ij}}$$

so that for uniaxial stress states the always positive effective stress $\bar{\sigma}$ has the value of the magnitude of the uniaxial stress that is being applied. The parameters Λ and m are material constants that could be temperature dependent, and ω is defined by

$$\omega = \begin{cases} 1 & \text{if } \bar{\sigma} \geq \sigma_c , \\ 0 & \text{if } \bar{\sigma} < \sigma_c \end{cases} \quad (39)$$

where σ_c is the uniaxial creep threshold stress for the material which will be temperature dependent

It is also postulated that

$$\frac{\partial \tilde{s}^I}{\partial t} = - P_I \tilde{\mu}^I \quad I = 1 \dots N,$$

$$\frac{\partial \hat{s}_{kl}^I}{\partial t} = - Q_I \hat{\mu}_{kl}^I , \quad I = 1 \dots N,$$

where P_I and Q_I , $I = 1 \dots N$ are material constants that may be temperature dependent. It should be noted that the form of (40) does not involve any interactions between the various internal state variables. The substitution of (37) and (40) into the inequalities (36) leads to

$$\sum_{I=1}^N P_I (\tilde{\mu}^I)^2 \geq 0 , \quad \frac{2}{3} \Lambda \omega \bar{\sigma}^{2m+2} + \sum_{I=1}^N Q_I \mu_{kl}^I \mu_{kl}^I \geq 0 .$$

The inequalities (41), ensuring that the second law of thermodynamics is obeyed, are satisfied for all possible states of the system provided that

$$\Lambda \geq 0, \quad P_I \geq 0, \quad Q_I \geq 0, \quad I = 1 \dots N$$

For states of equilibrium where the internal state variables are no longer time dependent, it follows from (40) that in such states the thermodynamic potential μ_{kl} is everywhere zero.

5. Plastic deformation for monotonic loading

Perhaps the most popular plastic deformation law is that for incremental plasticity where work-hardening is allowed (see for example Hill [3] for a full description and several alternative laws for the case of monotonic loading). The constitutive relation governing plastic deformation is written in the following form, involving time derivatives rather than increments of stress and strain,

$$\begin{aligned} \frac{\partial e_{kl}^p}{\partial t} &= \hat{\sigma}_{kl} \frac{\partial \lambda}{\partial t} \quad \text{if } \bar{\sigma} = F(W_p) \text{ or } W_p = \phi(\bar{\sigma}) \\ \frac{\partial e_{kl}^p}{\partial t} &= 0, \quad \text{if } \bar{\sigma} < F(W_p) \end{aligned} \quad (43)$$

The parameter W_p is the total energy per unit volume that has been dissipated by plastic flow given by

$$W_p(t) = \int_{-\infty}^t \hat{\sigma}_{ij}(\tau) \frac{\partial e_{ij}^p(\tau)}{\partial \tau} d\tau$$

The expression (44) has made use of the fact that $e_{ii}^p = 0$ and it should be noted that the plastic strain rate will be zero for some time interval in the range defined by the integral in (44). The monotonic increasing function F of the energy dissipated W_p defines the work hardening behaviour, and is such that $F(0) = \sigma_Y$ where σ_Y is the initial uniaxial yield stress of the material. The inverse form of this relation is also monotonic increasing, and defined by the function f such that $f(\sigma_Y) = 0$. The parameter λ is eliminated from the constitutive law (43) by the imposition of the yield criterion $\bar{\sigma} = F(W_p)$ or its inverse $W_p = f(\bar{\sigma})$ as follows.

On using (38), (43) and (44)

$$W_p(t) = \frac{2}{3} \int_{-}^t \bar{\sigma}^2(\tau) \frac{\partial \lambda(\tau)}{\partial \tau} d\tau, \quad (45)$$

so that on differentiation for a state of yielding,

$$\frac{\partial W_p}{\partial t} = \frac{2}{3} \bar{\sigma}^2 \frac{\partial \lambda}{\partial t}$$

The inverse form of the yield criterion given in (43) asserts that

$$\frac{\partial W_p}{\partial t} = f'(\bar{\sigma}) \frac{\partial \bar{\sigma}}{\partial t}$$

and it then follows from (46) that during yielding arising from monotonic loading

$$\frac{\partial \lambda}{\partial t} = \frac{3}{2} \frac{f'(\bar{\sigma})}{\bar{\sigma}^2} \frac{\partial \bar{\sigma}}{\partial t}$$

Thus the incremental plasticity law may now be written in the following form that does not involve the parameter λ , nor the energy dissipated W_p

$$\frac{\partial e_{kl}^p}{\partial t} = \frac{3}{2} \hat{\sigma}_{kl} \frac{f'(\bar{\sigma})}{\bar{\sigma}^2} \frac{\partial \bar{\sigma}}{\partial t} \quad \text{if } \frac{\partial \bar{\sigma}}{\partial t} \geq 0$$

(49)

$$\frac{\partial e_{kl}^p}{\partial t} = 0, \quad \text{if } \frac{\partial \bar{\sigma}}{\partial t} < 0$$

It follows from (38) and (49) that

$$\sigma_{kl} \frac{\partial e_{kl}^p}{\partial t} = f'(\bar{\sigma}) \frac{\partial \bar{\sigma}}{\partial t} = \frac{\partial W_p}{\partial t}, \quad \text{if } \frac{\partial \bar{\sigma}}{\partial t} \geq 0$$

$$\sigma_{kl} \frac{\partial e_{kl}^p}{\partial t} = 0, \quad \text{if } \frac{\partial \bar{\sigma}}{\partial t} < 0.$$

Since W_p is a non-decreasing function of the time t , it is clear that the constitutive law (49) is consistent with the second of the inequalities (36) demanded by the second law of thermodynamics.

An effective scalar plastic strain rate may be defined by

$$\frac{\partial \bar{\epsilon}_p}{\partial t} = \sqrt{\frac{2}{3} \frac{\partial e_{ij}^p}{\partial t} \frac{\partial e_{ij}^p}{\partial t}}$$

On substituting (49) into (51), it can then be shown that

$$\frac{\partial \bar{\epsilon}_p}{\partial t} = \frac{f'(\bar{\sigma})}{\bar{\sigma}} \frac{\partial \bar{\sigma}}{\partial t}$$

The effective scalar plastic strain during monotonic loading is then defined by

$$\bar{\epsilon}_p = \int_{-\infty}^t \frac{\partial \bar{\epsilon}_p(\tau)}{\partial \tau} d\tau = \int_0^{\bar{\sigma}} \frac{f'(\sigma)}{\sigma} d\sigma \quad (53)$$

It should be noted that the definition of $\bar{\epsilon}_p$ is such that it is always positive and it corresponds to the magnitude of the uniaxial effective plastic strain when the stress state is uniaxial.

As an example it is now assumed that the plastic strain - stress relation for monotonic loading has the following power law form

$$\bar{\epsilon}_p = \begin{cases} \epsilon_0 \left(\frac{\bar{\sigma}}{\sigma_Y} - 1 \right) & \text{if } \bar{\sigma} \geq \sigma_Y, \\ 0 & \text{if } \bar{\sigma} < \sigma_Y, \end{cases} \quad (54)$$

where ϵ_0 and n are material constants, and where σ_Y is the initial yield stress. Such a law is consistent with (53) only if the function f defining the work hardening has the form

$$f(\bar{\sigma}) \equiv f(\bar{\sigma}, \sigma_Y, \epsilon_0) = W_p = \frac{\sigma_Y \epsilon_0}{n+1} \left(\frac{\bar{\sigma}}{\sigma_Y} - 1 \right)^n \left[n \frac{\bar{\sigma}}{\sigma_Y} + 2n + 1 \right], \quad \bar{\sigma} \geq \sigma_Y, \quad (55)$$

having derivative given by

$$f'(\bar{\sigma}, \sigma_Y, \epsilon_0) = \begin{cases} n \epsilon_0 \frac{\bar{\sigma}}{\sigma_Y} \left(\frac{\bar{\sigma}}{\sigma_Y} - 1 \right)^{n-1} & \text{if } \bar{\sigma} \geq \sigma_Y, \\ 0 & \text{if } \bar{\sigma} < \sigma_Y \end{cases} \quad (56)$$

6. The constitutive equations for combined plasticity and creep

Having discussed the constitutive equations for both creep and plastic deformation separately, it is now appropriate to combine them into a single set of relations. It follows from (5), (33), (37) and (49) that

$$\tilde{\epsilon} = 3\alpha(\theta - \theta_0) + \frac{-2\nu}{E}\tilde{\sigma} - \sum_{I=1}^N A_I \tilde{s}$$

$$\frac{\partial \hat{\epsilon}_{kl}}{\partial t} = \frac{\partial}{\partial t} \left[\frac{1+\nu}{E} \hat{\sigma}_{kl} - \sum_{I=1}^N C_I \hat{s}_{kl}^I \right] + \left[\Lambda \omega \bar{\sigma}^{2m} + \frac{3}{2} \frac{f'(\bar{\sigma}, \sigma_Y, \epsilon_0)}{\bar{\sigma}^2} \frac{\partial \bar{\sigma}}{\partial t} \right] \hat{\sigma}_{kl}$$

where $\bar{\sigma}$ is defined by (38). The time derivatives of the internal state variables that appear in (57) are derived from (34) and (40), leading to the differential equations

$$\frac{\partial \tilde{s}^I}{\partial t} = -P_I (A_I \tilde{\sigma} + B_I \tilde{s}^I), \quad I = 1 \dots N,$$

$$\frac{\partial \hat{s}_{kl}^I}{\partial t} = -Q_I (C_I \hat{\sigma}_{kl} + D_I \hat{s}_{kl}^I), \quad I = 1 \dots N.$$

For the special case where the constants $P_I, Q_I, A_I, B_I, C_I, D_I, I = 1 \dots N$, are independent of time the differential equations may be solved leading to the results

$$\tilde{s}^I(t) = \frac{A_I}{B_I} e^{-B_I P_I t} \int_0^t e^{B_I P_I \tau} \frac{\partial \tilde{\sigma}(\tau)}{\partial \tau} d\tau - \tilde{\sigma}(t) \quad I = 1 \dots N$$

$$\hat{s}_{kl}^I(t) = \frac{C_I}{D_I} e^{-D_I Q_I t} \int_0^t e^{D_I Q_I \tau} \frac{\partial \hat{\sigma}_{kl}(\tau)}{\partial \tau} d\tau - \hat{\sigma}_{kl}(t) \quad I = 1 \dots N$$

On integrating by parts the results (59) and (60) may also be written in the following form which is very useful when the stress components are discontinuous in time

$$\tilde{s}^I(t) = -A_I P_I e^{-B_I P_I t} \int_0^t e^{B_I P_I \tau} \tilde{\sigma}(\tau) d\tau \quad I = 1 \dots N, \quad (61)$$

$$\hat{s}_{kl}^I(t) = -C_I Q_I e^{-D_I Q_I t} \int_0^t e^{D_I Q_I \tau} \hat{\sigma}_{kl}(\tau) d\tau \quad I = 1 \dots N$$

where use has been made of the fact that the components of the stress tensor are all zero at time $t = 0$.

It is convenient to introduce the following parameters in order to reduce to a minimum the number of independent parameters controlling stress-strain behaviour

$$k_I = B_I P_I, \quad K_I = D_I Q_I, \quad U_I = A_I^2 P_I \geq 0, \quad V_I = C_I^2 Q_I \geq 0, \quad I = 1 \dots N \quad (63)$$

The substitution of (61) and (62) into (57) then leads to the following relations governing the stress-strain response of the material

$$\tilde{\epsilon}(t) = 3\alpha(\theta - \theta_0) + \frac{1-2\nu}{E} \tilde{\sigma}(t) + \sum_{I=1}^N U_I e^{-k_I t} \int_0^t e^{k_I \tau} \tilde{\sigma}(\tau) d\tau, \quad (64)$$

$$\begin{aligned} \frac{\partial \hat{\epsilon}_{kl}(t)}{\partial t} = & \frac{1+\nu}{E} \frac{\partial \hat{\sigma}_{kl}(t)}{\partial t} + \left(\sum_{I=1}^N V_I \right) \hat{\sigma}_{kl}(t) - \sum_{I=1}^N V_I K_I e^{-K_I t} \int_0^t e^{K_I \tau} \hat{\sigma}_{kl}(\tau) d\tau \\ & + \left[\Lambda \omega \bar{\sigma}^{2m}(t) + \frac{3}{2} \frac{f'(\bar{\sigma}(t), \sigma_Y, \epsilon_0)}{\bar{\sigma}^2(t)} \frac{\partial \bar{\sigma}(t)}{\partial t} \right] \hat{\sigma}_{kl}(t) \end{aligned} \quad (65)$$

The corresponding values of the internal state variables are then given by

$$\tilde{s}^I(t) = -\frac{A_I k_I}{B_I} e^{-k_I t} \int_0^t e^{k_I \tau} \tilde{\sigma}(\tau) d\tau \quad I = 1 \dots N, \quad (66)$$

$$\hat{s}_{kl}^I(t) = -\frac{C_I K_I}{D_I} e^{-K_I t} \int_0^t e^{K_I \tau} \hat{\sigma}_{kl}(\tau) d\tau \quad I = 1 \dots N \quad (67)$$

It should be noted that the values of the time constants k_I and K_I have the same signs as B_I and D_I respectively because of the result (42). When B_I and D_I are both positive it follows that k_I and K_I will also be positive leading to internal state variables that always return to zero when all the stress components are reduced to zero some time after loading.

7. Discretisation of the stress-strain relations

The stress-strain equations (64) and (65) can be solved analytically for special cases, but for general loading conditions leading to homogeneous time dependent stress states it is necessary to predict stress-strain behaviour using numerical methods. This is achieved by introducing a small time step Δt and discretising the stress-strain equations. The applied loading, whether stress or strain controlled as assumed to be cyclic having a frequency n . The time step Δt is defined so that $\Delta t = 1/pn$ where p

is the number of time intervals per cycle. Define an equally spaced sequence of times $t_j = j\Delta t$, $j = 1, 2, \dots$, and for any time dependent function $g(t)$ let $g_j = g(t_j)$.

The stress-strain relations (64) and (65) are now approximated by

$$\begin{aligned}\tilde{\epsilon}_{j+1} &= 3\alpha(\theta - \theta_0) + \frac{1-2\nu}{E}\tilde{\sigma}_{j+1} + \Delta t \sum_{I=1}^N U_I e^{-k_{ij}\Delta t} \left(\sum_{i=0}^j \delta_i^j e^{k_{ij}\Delta t} \tilde{\sigma}_i \right) \quad j = 0, 1, 2, \dots, \\ \hat{\epsilon}_{kl}^j &= \hat{\epsilon}_{kl}^{j-1} + \frac{1+\nu}{E}(\hat{\sigma}_{kl}^j - \hat{\sigma}_{kl}^{j-1}) + \Delta t \left(\sum_{I=1}^N V_I \right) \hat{\sigma}_{kl}^j - (\Delta t)^2 \sum_{I=1}^N V_I K_I e^{-K_{ij}\Delta t} \left(\sum_{i=0}^j \delta_i^j e^{K_{ij}\Delta t} \hat{\sigma}_{kl}^i \right) \\ &\quad + \Lambda \omega \bar{\sigma}_j^{2m} \hat{\sigma}_{kl}^j \Delta t + \Phi_{kl}^j \quad j = 1, 2, \dots, \quad (69)\end{aligned}$$

where

$$\delta_i^j = \begin{cases} \frac{1}{2} & \text{if } i=0 \text{ or } i=j \\ 1 & \text{if } 1 \leq i \leq j-1 \end{cases}$$

and where for the case of initial loading to the point of the first occurrence of elastic unloading

$$\Phi_{kl}^j = \frac{3}{2} \frac{f'(\bar{\sigma}_j, \sigma_Y, \epsilon_0)}{\bar{\sigma}_j^2} (\bar{\sigma}_j - \bar{\sigma}_{j-1}) \hat{\sigma}_{kl}^{j-1}.$$

The trapezoidal integration rule has been used when discretising the integrals appearing in the stress-strain equations (64) and (65). As the initial stress field is zero everywhere, the following initial stress values are specified

$$\tilde{\sigma}_0 = 0, \quad \hat{\sigma}_{kl}^0 = 0, \quad \bar{\sigma}_0 = 0$$

The corresponding initial strain values are given by

$$\tilde{\epsilon}_0 = 3\alpha(\theta - \theta_0), \quad \hat{\epsilon}_{kl}^0 = 0$$

which are all zero only if the temperature remains at its reference value. The discrete forms of (66) and (67) are given by

$$\tilde{s}_j^I = -\frac{A_I}{B_I} k_I \Delta t e^{-k_{ij}\Delta t} \sum_{i=0}^j \delta_i^j e^{k_{ij}\Delta t} \tilde{\sigma}_i, \quad j \geq 1, \quad I = 1 \dots N$$

$$\hat{s}_{jkl}^I = -\frac{C_1}{D_1} K_I \Delta t e^{-K_{ij} \Delta t} \sum_{i=0}^j \delta_i^j e^{K_{ij} \Delta t} \hat{\sigma}_{kl}^i, \quad j \geq 1, \quad I = 1 \dots N, \quad (75)$$

where the initial values are given by $\tilde{s}_0^I = \hat{s}_{0kl}^I = 0$ since the internal state variables have been defined to be zero in the reference state.

8. Plasticity during unloading and re-loading

The best approach for the modelling of plastic flow during unloading and subsequent re-loading, as would arise during cyclic loading, is not obvious from an examination of the literature, nor from the theory manuals used by FEA packages such as ABAQUS and ANSYS. There are a variety of types of behaviour that are available when carrying out FEA, for example :

- Bi-linear kinematic
- 2. Multi-linear kinematic
- 3. Bi-linear isotropic
- 4. Multi-linear isotropic

Isotropic hardening models assume that on monotonic uniaxial loading to an applied stress value σ_Y the material behaves elastically. Further load increases lead to work hardening to a stress value $\sigma_{\max}$. On unloading the material remains elastic until the uniaxial load attains the value $-\sigma_{\max}$. Kinematic hardening models assume that on uniaxial loading to an applied stress value σ_Y the material behaves elastically. Further load increases lead to work hardening to a stress value $\sigma_{\max}$. On unloading the material remains elastic until the uniaxial load attains the value $\sigma_{\max} - 2\sigma_Y$. The plasticity modelling to be assumed here will be based on kinematic hardening which is thought to be the more realistic model.

When using kinematic hardening models for applied loads which are not monotonic increasing, it is important to identify the occurrences of elastic unloading or re-loading. The stress-strain behaviour following the onset of elastic deformation is most easily modelled by modifying the relations determining the plastic flow from the point of onset of any elastic deformation whether unloading or re-loading. Following the initial loading stage, for any change from plastic to elastic deformation or vice versa the relation (71) is replaced by

$$\Phi_{kl}^j = \frac{3}{2} \frac{f'(\Delta\bar{\sigma}_j, 2\sigma_Y, 2\varepsilon_0)}{\Delta\bar{\sigma}_j^2} (\Delta\bar{\sigma}_j - \Delta\bar{\sigma}_{j-1}) (\hat{\sigma}_{kl}^j - \hat{\sigma}_{kl}^e)$$

where $\Delta\bar{\sigma}$ denotes the difference $\bar{\sigma} - \bar{\sigma}_e$ where $\bar{\sigma}_e$ denotes the last value of $\bar{\sigma}$ for which there was a transition between elastic and plastic deformation or vice versa. The parameter $\hat{\sigma}_{kl}^e$ is the value of $\hat{\sigma}_{kl}$ corresponding to $\bar{\sigma}_e$. It should be noted that (76) is identical to (71) except that $\bar{\sigma}_j$ and σ_Y have been replaced respectively by $\Delta\bar{\sigma}_j$ and $2\sigma_Y$, and ε_0 has been replaced by $2\varepsilon_0$. The replacement of the initial yield stress σ_Y by $2\sigma_Y$ is characteristic of models where reversed plasticity or internal friction are encountered. The corresponding replacement of ε_0 by $2\varepsilon_0$ is also made here in order that the term defined by (76) generates closed hysteresis loops for the case of cyclic loading. The relation (76) is a model for reversed plasticity that corresponds to an assumption of kinematic hardening.

9. Strain controlled deformation

In practice most experimental data are obtained under strain control. The model developed above, while most easily applied for situations of stress control, will now be configured so that it can be applied for the case of strain control. Consider first of all the stress-strain relation (64) which is rewritten in the form

$$\frac{1-2\nu}{E} \tilde{\sigma}(t) = \tilde{\varepsilon}(t) - 3\alpha(\theta - \theta_0) - \sum_{l=1}^N U_l e^{-k_l t} \int_0^t e^{k_l \tau} \tilde{\sigma}(\tau) d\tau,$$

enabling the stress $\tilde{\sigma}(t)$ at time t to be calculated from the controlled strain $\tilde{\varepsilon}(t)$ and the stress history $\tilde{\sigma}(\tau) : 0 \leq \tau < t$.

The stress-strain relation (65) is now rewritten in the form

$$\begin{aligned} \frac{1+\nu}{E} \frac{\partial \hat{\sigma}_{kl}(t)}{\partial t} &= \frac{\partial \hat{\varepsilon}_{kl}(t)}{\partial t} - \left(\sum_{l=1}^N V_l \right) \hat{\sigma}_{kl}(t) + \sum_{l=1}^N V_l K_l e^{-K_l t} \int_0^t e^{K_l \tau} \hat{\sigma}_{kl}(\tau) d\tau \\ &\quad \left[\Lambda \omega \bar{\sigma}^{2m}(t) + \frac{3}{2} \frac{f'(\bar{\sigma}(t), \sigma_Y, \varepsilon_0)}{\bar{\sigma}^2(t)} \frac{\partial \bar{\sigma}(t)}{\partial t} \right] \hat{\sigma}_{kl}(t), \end{aligned} \tag{78}$$

determining the value of the current rate of change of the stress $\hat{\sigma}_{kl}$ in terms of the specified current strain rate, the rate of change of the effective stress $\bar{\sigma}$ and the stress history $\hat{\sigma}_{kl}(\tau) : 0 \leq \tau < t$. A method is needed for calculating the rate of change of the effective stress $\bar{\sigma}$ appearing in the last term of (78). One approach is to multiply the relation (78) by $\hat{\sigma}_{kl}(t)$ using the differentiated form of the definition (38), namely

$$\bar{\sigma} \frac{\partial \bar{\sigma}}{\partial t} = \frac{3}{2} \hat{\sigma}_{kl} \cdot \frac{\partial \hat{\sigma}_{kl}}{\partial t} \quad (79)$$

It can be shown that

$$\left[\frac{2(1+\nu)}{3E} \bar{\sigma}(t) + f'(\bar{\sigma}(t), \sigma_Y, \epsilon_0) \right] \frac{\partial \bar{\sigma}}{\partial t} = \quad (80)$$

$$\hat{\sigma}_{kl}(t) \left[\frac{\partial \hat{\sigma}_{kl}(t)}{\partial t} - \left(\sum_{l=1}^N V_l \right) \hat{\sigma}_{kl}(t) + \sum_{l=1}^N V_l K_l e^{-K_l t} \int_0^t e^{K_l \tau} \hat{\sigma}_{kl}(\tau) d\tau \right] - \frac{2}{3} \Lambda \omega \bar{\sigma}^{2m+2}(t),$$

determining the value of the rate of change of $\bar{\sigma}$ in terms of the current value of $\bar{\sigma}$, the specified strain rate and the stress history $\hat{\sigma}_{kl}(\tau) : 0 \leq \tau < t$. The value of the rate of change of $\bar{\sigma}$ predicted by (80) is substituted into (78) in order to estimate the rate of change of the stress components $\hat{\sigma}_{kl}$

The discrete forms of (77), (78) and (80) used for numerical calculations are given by, for values $\geq$

$$\frac{1-2\nu}{E} \bar{\sigma} = \tilde{e}_{j+1} - 3\alpha(\theta - \theta_0) - \Delta t \sum_{l=1}^N U_l e^{-K_{lj}\Delta t} \sum_{i=0}^j \delta_i^j e^{K_{lj}\Delta t} \tilde{\sigma}_i, \quad (81)$$

$$\frac{1+\nu}{E} \delta \hat{\sigma}_{kl}^{j+1} = \delta \hat{e}_{kl}^{j+1} - \Delta t \left(\sum_{l=1}^N V_l \right) \hat{\sigma}_{kl}^j + (\Delta t)^2 \sum_{l=1}^N V_l K_l e^{-K_{lj}\Delta t} \sum_{i=0}^j \delta_i^j e^{K_{lj}\Delta t} \hat{\sigma}_{kl}^i - \Lambda \omega \bar{\sigma}_j^{2m} \hat{\sigma}_{kl}^j \Delta t - \Psi_{kl}^j \delta \bar{\sigma}_{j+1} \quad (82)$$

$$\left[\frac{2(1+\nu)}{3E} \bar{\sigma}_j + \Psi_j \right] \delta \bar{\sigma}_{j+1} = \hat{\sigma}_{kl}^j \delta \hat{e}_{kl}^{j+1} \quad (83)$$

$$\frac{2\Delta t}{3} \sum_{l=1}^N V_l \bar{\sigma}_j^2 + \hat{\sigma}_{kl}^j (\Delta t)^2 \sum_{l=1}^N V_l K_l e^{-K_{lj}\Delta t} \sum_{i=0}^j \delta_i^j e^{K_{lj}\Delta t} \hat{\sigma}_{kl}^i - \frac{2}{3} \Lambda \omega \bar{\sigma}_j^{2m+2}$$

where

$$\Psi_j = f'(\bar{\sigma}_j, \sigma_Y, \varepsilon_0) , \quad \Psi_{kl}^j = \frac{3}{2} \frac{f'(\bar{\sigma}_j, \sigma_Y, \varepsilon_0)}{\bar{\sigma}_j^2} \hat{\sigma}_{kl}^j$$

$$\delta \hat{\sigma}_{kl}^{j+1} = \hat{\sigma}_{kl}^{j+1} - \hat{\sigma}_{kl}^j , \quad \delta \bar{\sigma}_{j+1} = \bar{\sigma}_{j+1} - \bar{\sigma}_j , \quad \delta \hat{e}_{kl}^{j+1} = \hat{e}_{kl}^{j+1} - \hat{e}_{kl}^j$$

The R.H.S.s of (81-83) may be calculated from the current value of the strain rate and the stress history, leading to estimates of the current stress when use is also made of (84).

Following the initial loading stage, for any change from plastic to elastic deformation or vice versa the relation (84) is replaced by

$$\Psi_i = (\Delta \bar{\sigma}_j, 2\sigma_Y, 2\varepsilon_0) , \quad \Psi_{kl}^j = \frac{3}{2} \frac{f'(\Delta \bar{\sigma}_j, 2\sigma_Y, 2\varepsilon_0)}{\Delta \bar{\sigma}_i^2} (\hat{\sigma}_{kl}^j - \hat{\sigma}_{kl}^e)$$

10. Example predictions

Predictions based on the analysis presented above can be made only if the mathematical relationships derived are implemented through computer codes. In the first instance software has been written in FORTRAN 77 as it is the most convenient and widely used language that enables accurate coding. However, it becomes clear that when checking the models by carrying out sample predictions, that having some form of visualisation is highly desirable. The FORTRAN 77 software has, therefore, been modified into the programming language C/C++ so that a very user-friendly interface between the user and the code can be written, as exemplified by the computer application known as LOOP+ (see section 11 for details). In this report there are given the results of predictions for various situations that exhibit different types of material behaviour. They are given primarily to enable users of LOOP+ to check that their software is working correctly. The sample predictions carried out here are summarised in Table 1 where it is seen that thermal expansion effects are being neglected (i.e. α has been set to zero). In the Table the units of the parameters are not given (nor in the application LOOP+) but it is essential that stresses are measured in MPa and time in seconds.

Table 1 : Parameter values used when testing the software; 1 set of damage parameters

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7	Test 8
E (GPa)	200	200	200	200	200	200	200	200
ν	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3
α ($^{\circ}\text{C}$)	0	0	0	0	0	0	0	0
n	5	5	5	5	5	5	5	5
σ_Y (GPa)	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
ε_0 ($\times 10^2$)	0	10	10	10	10	10	10	10
m	2	2	2	2	2	2	2	2

Λ ($\times 10^{-6}$)	0	0	0	0	0	0.05	0.05	0
σ_c (GPa)	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
U ($\times 10^5$)	0	0	0	0.5	0.5	0.5	0.5	0.5
k	0.1	0.1	0.1	0.1	0.1	0.1	0.05	0.05
V ($\times 10^5$)	0	0	0.5	0	0.5	0.5	0.5	0.5
K	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
Cycle type	sine	sine	sine	sine	sine	sine	sine	sine
Frequency (Hz)	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
Range (MPa)	800	800	800	800	800	800	800	800
Mean (MPa)	0	0	0	0	0	0	10	10

It is seen from Table 1 that the sample predictions are labelled Test 1 to Test 8. For all tests the applied stress is controlled to be sinusoidal in nature such that the frequency of stress cycling and the stress range are set at 0.1 Hz and 800 MPa respectively. The mean applied stress for cycling is zero for Tests 1-6 and 10 MPa for Tests 7 and 8. The Test 1 results shown in Fig.1 are such that the material's response to cyclic stress-controlled loading is linear elastic where there is no hysteresis loss. This type of behaviour results when the parameters ϵ_0 , Λ , U and V are set to zero.

The Test 2 results are shown in Fig.2 which shows the corresponding response for elastic-plastic behaviour where repeated stress cycles generate identical stress-strain loops. For this case the parameters Λ , U and V are set to zero.

For Test 3 anelastic strain occurs that is governed by the shear stresses applied to the material (i.e. $\Lambda = 0$, $U = 0$ but $V \neq 0$). For Test 4 a corresponding anelastic strain occurs that is governed by the deviatoric stresses applied to the material (i.e. $\Lambda = 0$, $V = 0$ but $U \neq 0$), while for Test 5 anelastic strains arising from both deviatoric and shear stresses are considered material (i.e. $\Lambda = 0$, but $U \neq 0$ and $V \neq 0$). The results for Tests 3-5, showing the effect of anelastic strain, are shown in Figs. 3-5. It is observed that repeated stress cycling results in the migration of the stress-strain loops. Because of the need to use a large number of stress increments when carrying out stress cycling, and because of spreadsheet memory restrictions, only four complete stress-strain loops are shown in Figs. 3-5. The nature of anelastic strain is such that continued stress cycling leads to a steady state stress-strain loop where the same loop is generated once the migration effect becomes negligible. This effect is readily observed when using the LOOP+ application to generate the stress-strain cycles. The anelastic strain being modelled in Tests 3-5 could be relevant for room temperature deformation, but is definitely of great importance when considering high temperature deformation where it describes the reversible creep deformation that has occurred. Such deformation is reversible in the sense that the anelastic component of creep deformation will recover after a sufficient time to a zero value if the material is unloaded and held at the high temperature.

Test 6 includes an irreversible component of creep strain that will arise whenever the parameter Λ is non-zero. For this case continued stress cycling at zero mean load leads to a stable non-migrating stress-strain loop. The conditions for Test 7 are identical to those for Test 6 except that the constant k has been halved to the value 0.05. The conditions for Test 8 are identical to those for Test 7 except that the parameter Λ has been set to zero leading to a zero irreversible creep component of the axial strain.

11. The Loop+ Program

Introduction

The Loop+ program runs under Windows 95, and provides a simple interface to the mathematical model described in this document. Because of the complexity of the model, it was necessary to produce this program early on in the development cycle - the interface and the model were developed in tandem during the later parts of the project, the program being used to provide results which were fed back into the development and verification of the model. It provides almost instant feedback when materials parameters or stress waveforms are altered by the user, allowing them to quickly work through a number of "what-if" options.

Current state of the program

The rest of this section describes the user interface for the program, and gives simple instructions on its use. However, it must be pointed out here that the program is not yet finished, and certain parts of the user interface. Our primary focus at this point has been the stability of the program and the model, and there are some usability issues which will have to be cleared up before the program is finished.

Using the program

Material Properties

The Loop+ program is arranged in four sections, using a tabbed-page interface. Upon running the program, you will see the first page "Material Properties". This allows the user to enter a name for the material, its parameters and damage parameters. A slider allows the user to specify how many sets of damage parameters will be entered, and next and previous buttons cycle through the damage parameter sets available. Below this, the material's other properties can be entered. Each parameter can detect whether the value in it is valid for the model, and give instant feedback if the value is

invalid. Tooltips (highlighted text boxes) appear to show valid ranges for the parameters. A Load and Save button in the bottom right allow loading or saving of materials parameters.

Loading Details

The next page shows the details of the loading that will be applied to the specimen. First of all the user can choose whether to perform the experiment under stress or strain control. Below that is a list of the loadings - three sets of physical loadings, and a temperature one. At the moment, only the first loading is relevant, although the model can cope with more loadings and this will be implemented in future versions of the program. Clicking on the first item in the list and pressing Modify (or double clicking the item) brings up a wave-form editor allowing the user to specify a constant, sinusoidal, or triangular wave. Other wave-forms can be specified through the “arbitrary” wave type, which allows a text file to provide the shape of the wave. The user can alter the frequency, mean, and range of the wave-form using the number fields on the right of the dialog. Returning to the Loading page, the user can use the Load and Save buttons in the bottom right to load or save wave-form details.

Options

The options page allows the user to set the miscellaneous options for the simulation: how many points per cycle (more points is slower but more accurate), how many cycles to run for, whether to save the data to a text file. It also allows the user to read in a previously saved text file to use as an overlay on a simulation, giving the option to rapidly compare the difference made by a parameter change. The Load and Save buttons here save the current options.

Run Simulation

The final page allows the user to run the simulation and see the results in a hysteresis loop plotted on the left hand of the screen. Spin buttons allow the user to adjust the scale of the plot to a level of zoom that allows them to distinguish the information they are interested in, and the plot automatically re-scales if any data points exceed its bounds. The user can pause and restart the simulation at any time, and can flick back to the other pages (which will allow them to view, but not alter the data while the simulation is running).

Other Features

The menu bar provides two more options. The Project menu allows the user to save and load all details (materials properties, wave-forms, and options) in a single file, ensuring that all relevant data is kept together. The Help menu provides information about the program. Although it is not yet implemented, it is intended that context-sensitive help will be available for all fields.

12. Suggestions for further work

The approach described in this report is the first step towards the development of a model that can predict combined fatigue and creep deformation in the presence of multi-axial stress states. There are various aspects that need further investigation. These are now listed :

In order to predict realistic elastic-plastic behaviour where cyclic loading at zero mean stress leads to transient stress-strain behaviour before steady state conditions are established, the elastic plastic analysis needs to be extended so that stress history effects are included. It is suggested that the treatment in this report of time dependent anelastic deformation is modified so that the role of time is replaced by the cumulative plastic work done. This is expected to lead to the observed transient effects without introducing any time dependence into the model.

2. The model that has been developed needs to be validated by comparison of predictions with experimental data. While it has been shown that the model can deal with both stress controlled cycling (triaxial) and for axial strain control for uniaxial loading, the software system has been developed only for stress controlled cycling. Some preliminary software for strain control under uniaxial loading conditions has been written but it has not yet been possible to establish robust performance. Further development of the software is needed to establish robust performance for the simulation of strain control during cycling

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Figure 1: First 4 loops of Test 1

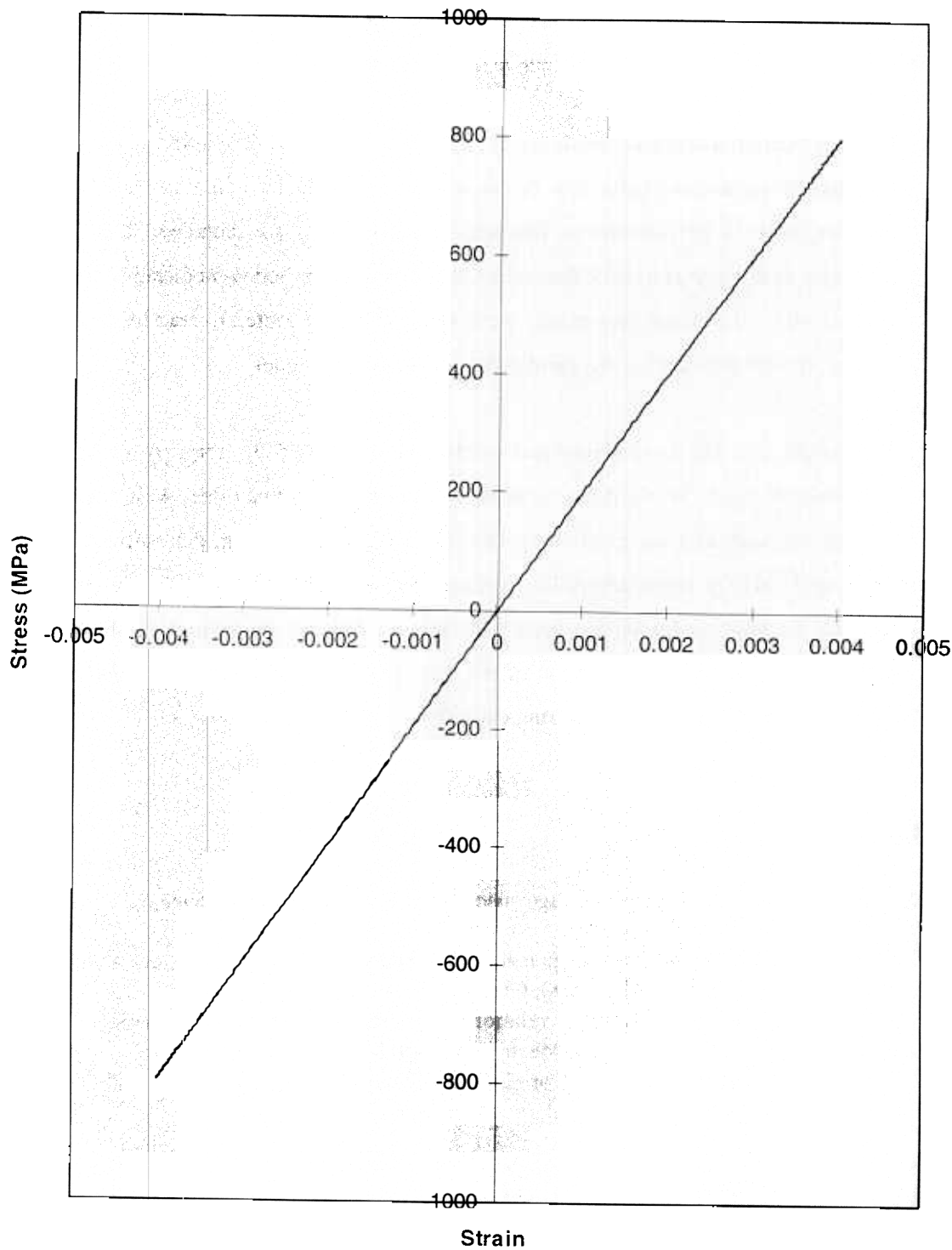


Figure 2: First 4 loops of Test 2

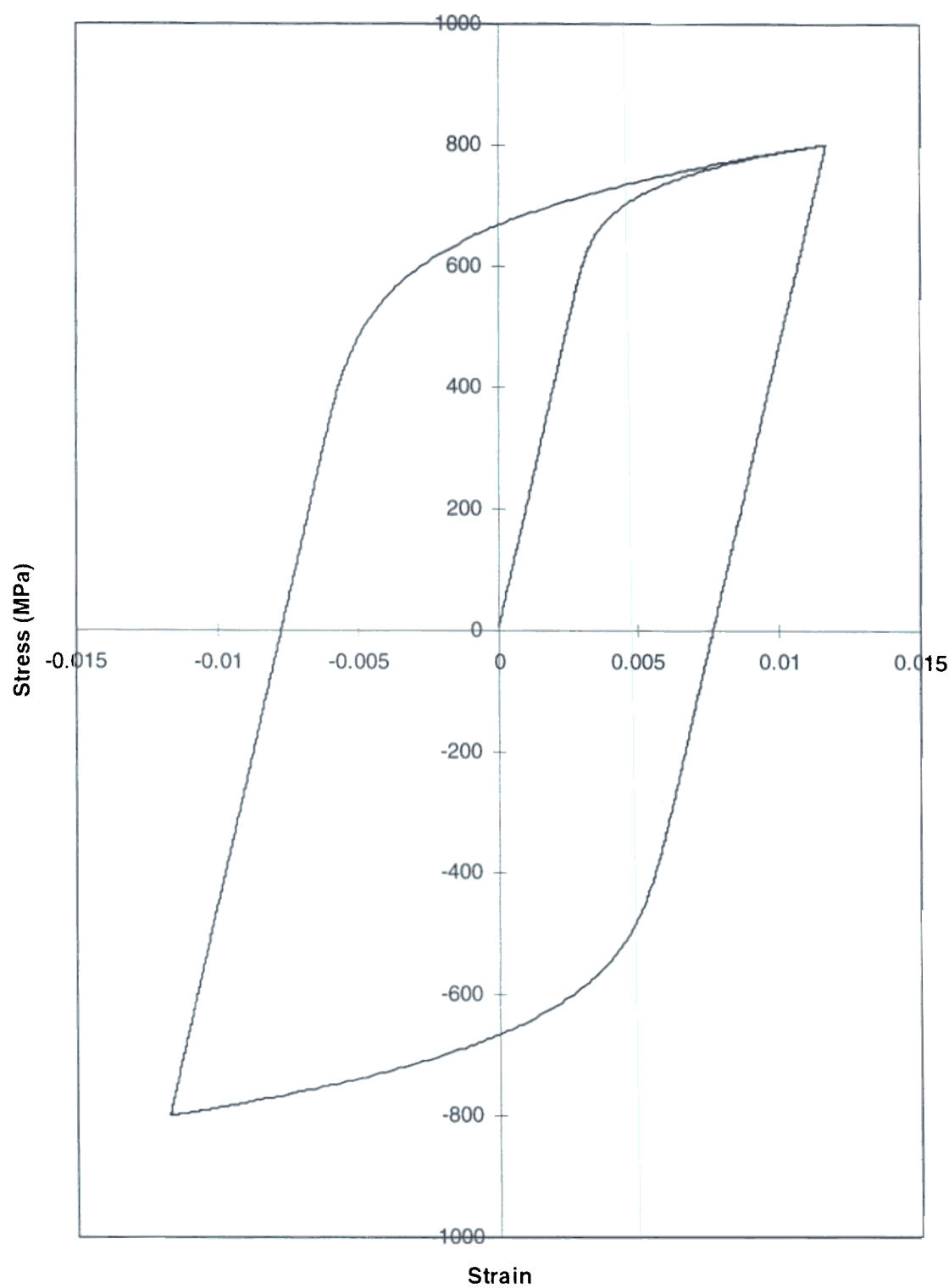


Figure 3: First 4 loops of Test 3

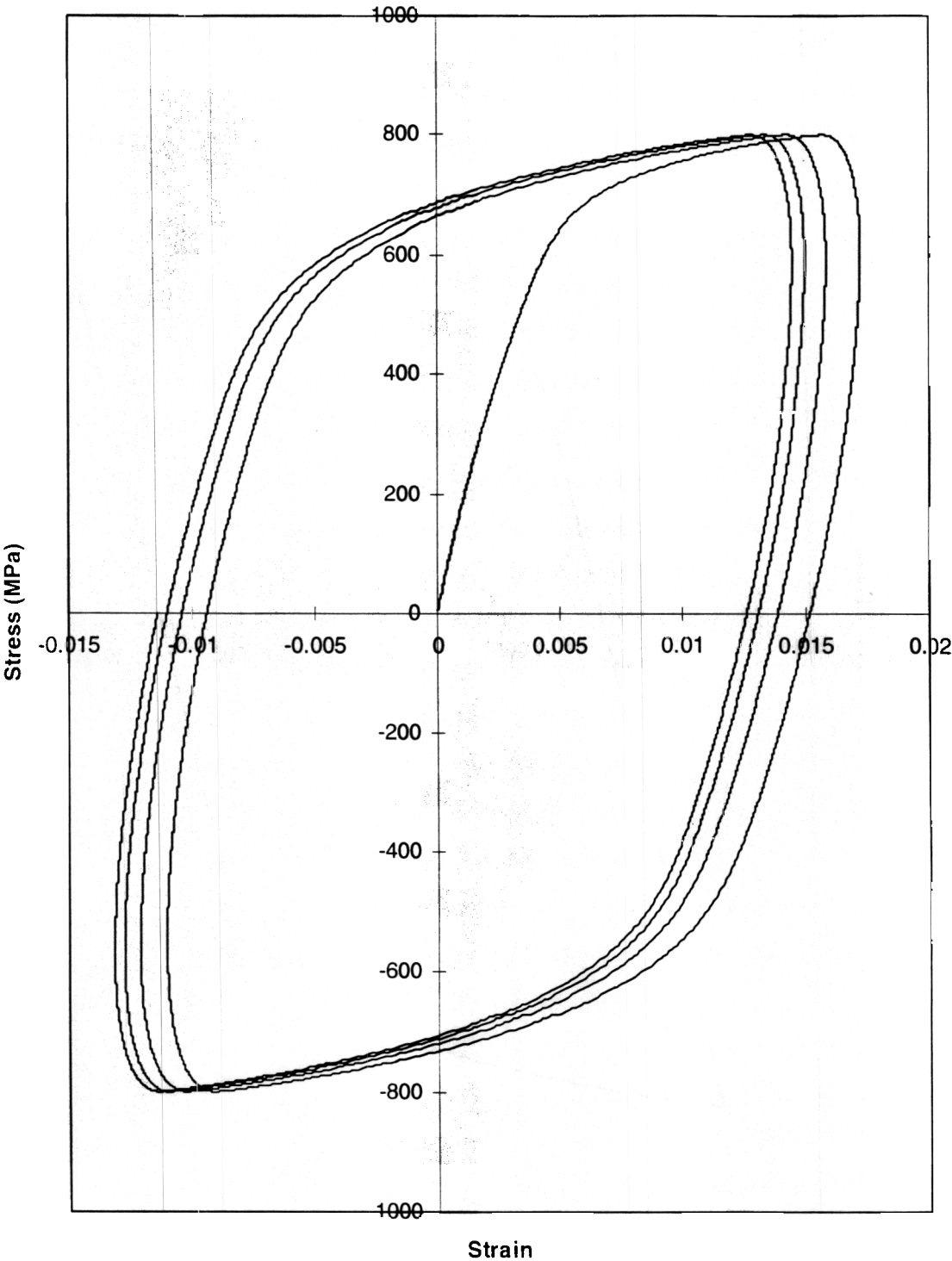


Figure 4: First 4 loops of Test 4

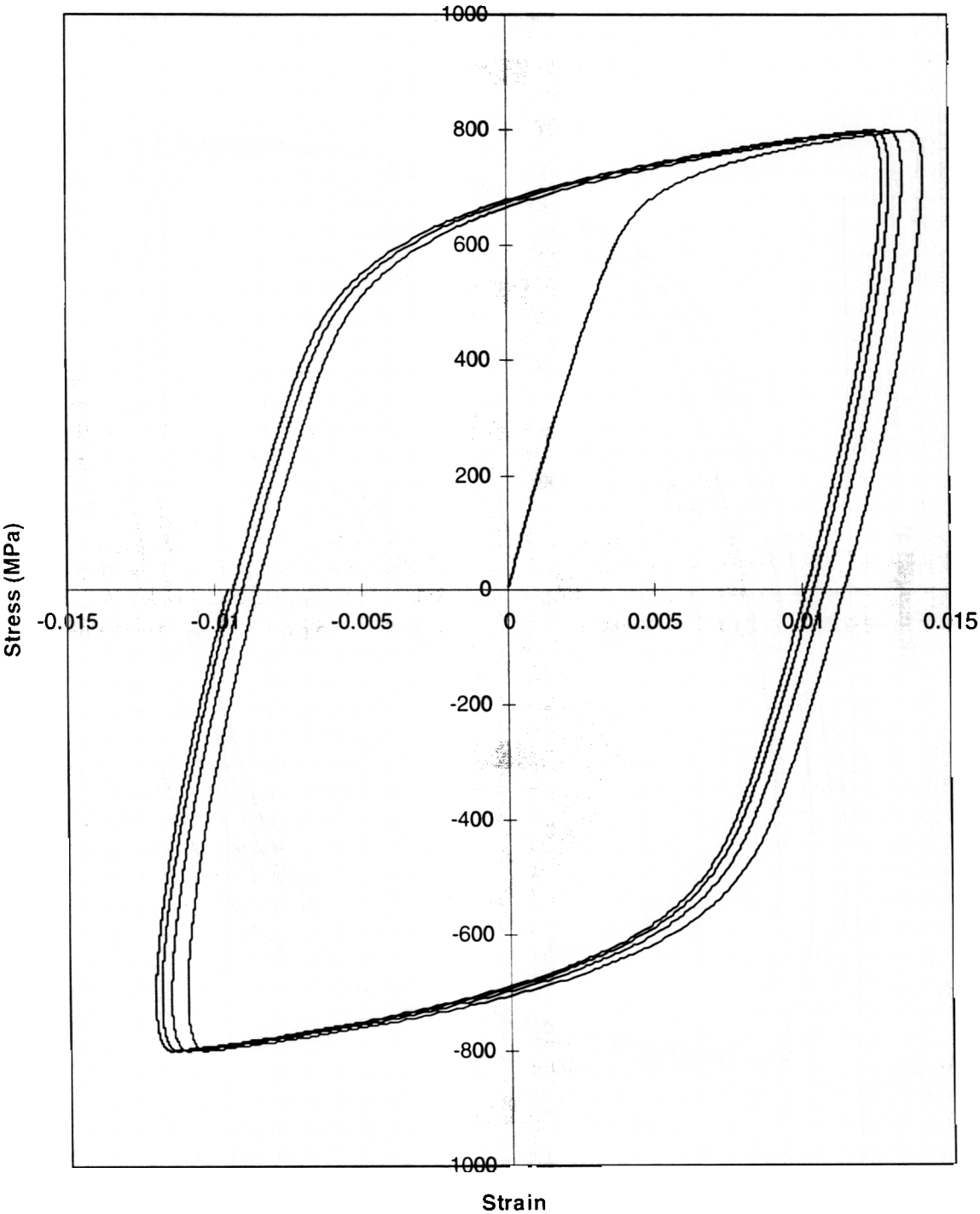


Figure 5: First 4 loops of Test 5

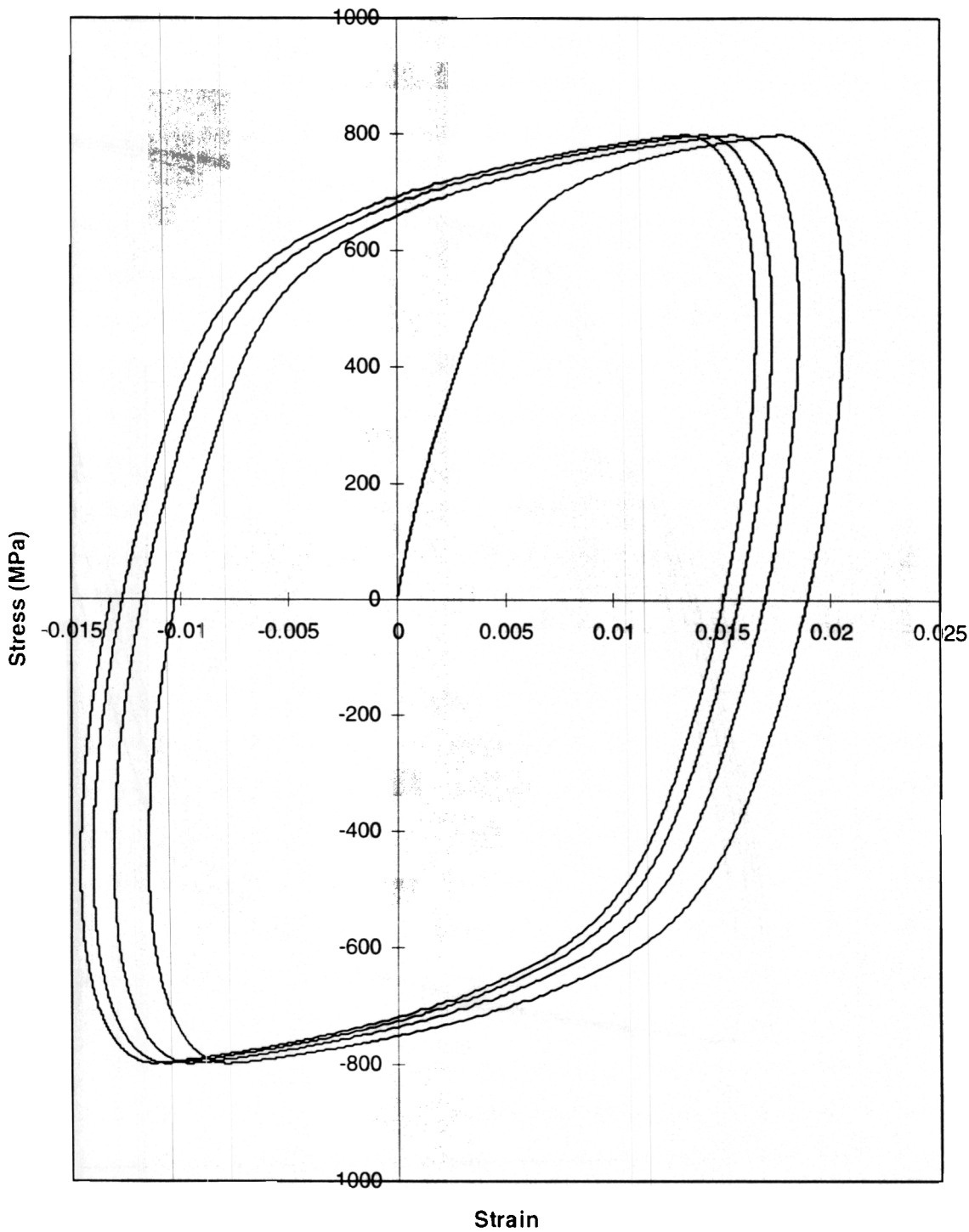


Figure 6: First 4 loops of Test 6

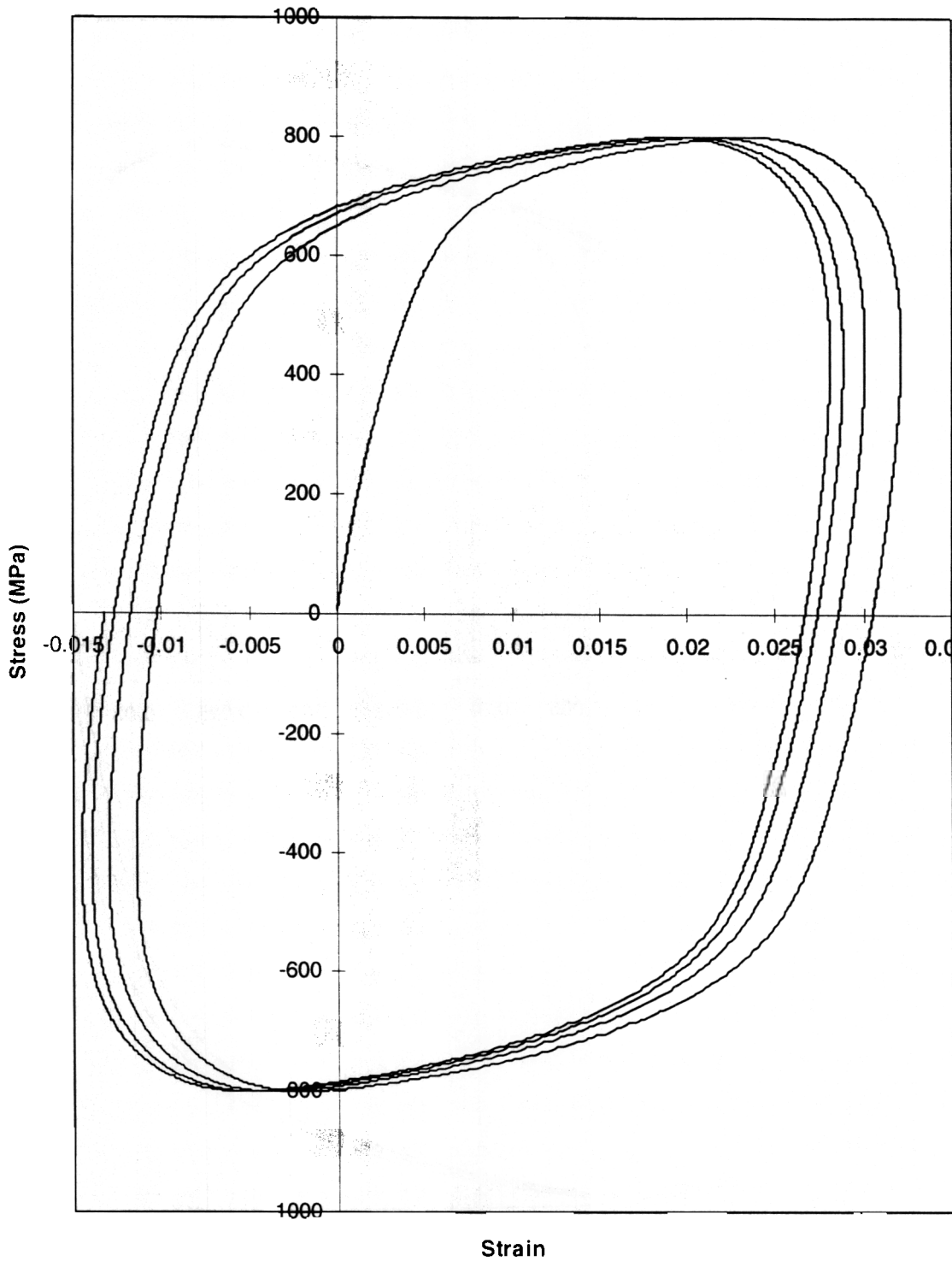


Figure 7: First 4 loops of Test 7

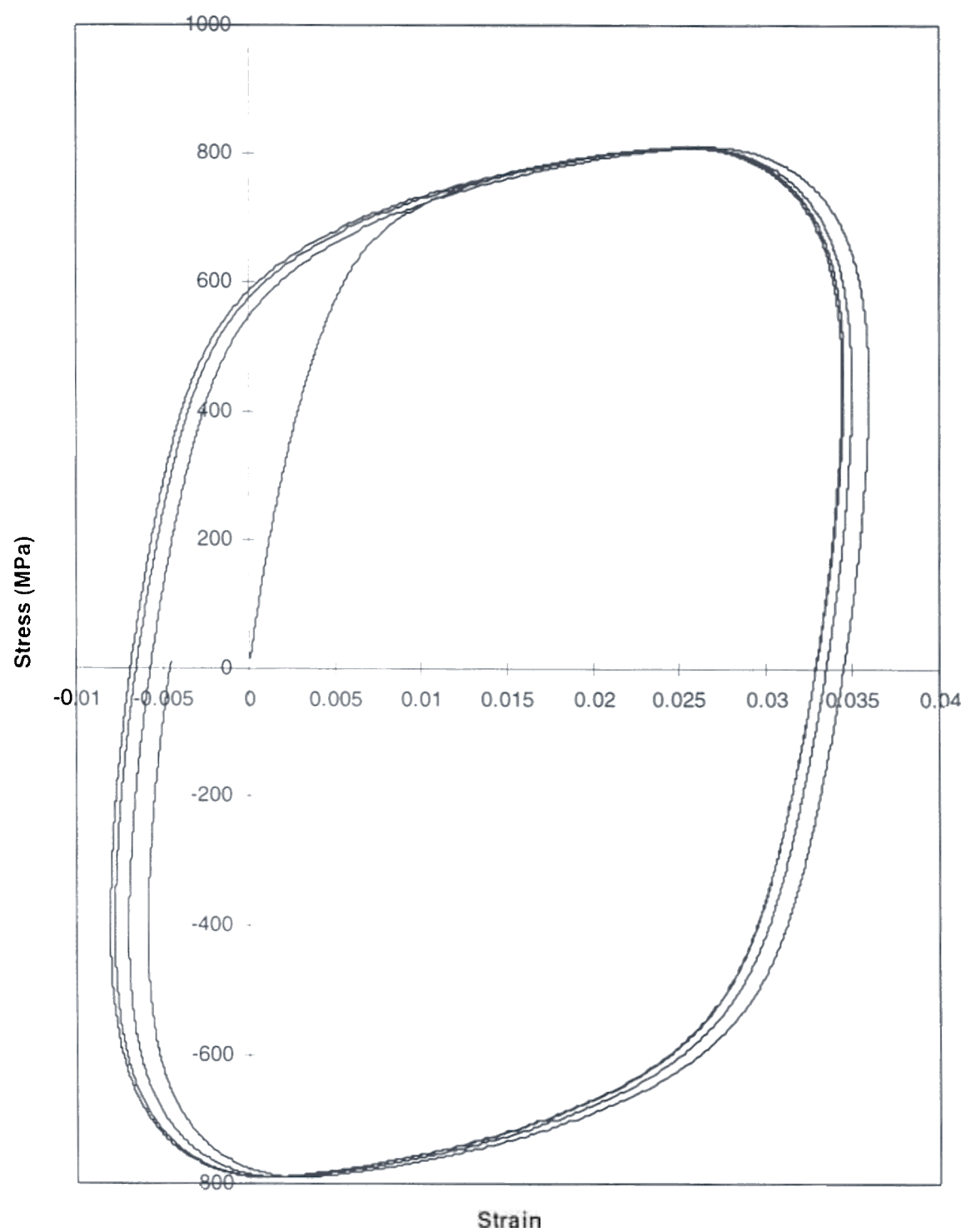


Figure 8: First 4 loops of Test 8

